

HOMOGENEOUS MODELS FOR LEVI DEGENERATE CR MANIFOLDS

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ABSTRACT. We extend the notion of a fundamental negatively \mathbb{Z} -graded Lie algebra $\mathfrak{m}_x = \bigoplus_{p \leq -1} \mathfrak{m}_x^p$ associated to any point of a Levi nondegenerate CR manifold to the class of k -nondegenerate CR manifolds $(M, \mathcal{D}, \mathcal{J})$ for all $k \geq 2$ and call this invariant the core at $x \in M$. It consists of a \mathbb{Z} -graded vector space $\mathfrak{m}_x = \bigoplus_{p \leq k-2} \mathfrak{m}_x^p$ of height $k-2$ endowed with the natural algebraic structure induced by the Tanaka and Freeman sequences of $(M, \mathcal{D}, \mathcal{J})$ and the Levi forms of higher order. In the case of CR manifolds of hypersurface type we propose a definition of an homogeneous model of type \mathfrak{m} , that is an homogeneous k -nondegenerate CR manifold $M = G/G_o$ with core \mathfrak{m} associated with an appropriate \mathbb{Z} -graded Lie algebra $Lie(G) = \mathfrak{g} = \bigoplus \mathfrak{g}^p$ and subalgebra $Lie(G_o) = \mathfrak{g}_o = \bigoplus \mathfrak{g}_o^p$ of the nonnegative part $\bigoplus_{p \geq 0} \mathfrak{g}^p$. It generalizes the classical notion of Tanaka of homogeneous model for Levi nondegenerate CR manifolds and the tube over the future light cone, the unique (up to local CR diffeomorphisms) maximally homogeneous 5-dimensional 2-nondegenerate CR manifold. We investigate the basic properties of cores and models and study the 7-dimensional CR manifolds of hypersurface type from this perspective. We first classify cores of 7-dimensional 2-nondegenerate CR manifolds up to isomorphism and then construct homogeneous models for seven of these classes. We finally show that there exists a unique core and homogeneous model in the 3-nondegenerate class.

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1. INTRODUCTION

A CR structure on a manifold M of dimension $m = 2d + c$ is a pair $(\mathcal{D}, \mathcal{J})$ given by a rank $2d$ distribution $\mathcal{D} \subset TM$ and a smooth family of complex structures $\mathcal{J}_x : \mathcal{D}|_x \rightarrow \mathcal{D}|_x$ with the

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property that the \mathcal{J} -eigenspace distribution $\mathcal{D}^{10} \subset T^{\mathbb{C}}M$ corresponding to the eigenvalue $+i$ is involutive. The integers d, c are respectively called the CR dimension and the codimension of the CR structure $(\mathcal{D}, \mathcal{J})$ and the complex vector bundles \mathcal{D}^{10} and $\mathcal{D}^{01} = \overline{\mathcal{D}^{10}}$ are the holomorphic and anti-holomorphic bundles. We refer the reader to e.g. [2, 7, 35] for a general introduction to CR manifolds.

The equivalence problem of CR manifolds, and the associated problem of constructing a full system of invariants which distinguish one CR manifold from another, is well-understood, at least in the case of strongly regular CR manifolds with *nondegenerate* Levi form (see [6, 31, 32]). We recall that the Levi form of a CR manifold $(M, \mathcal{D}, \mathcal{J})$ is the \mathcal{J} -Hermitian skew-symmetric bundle map

$$\mathcal{L} : \mathcal{D} \times \mathcal{D} \longrightarrow TM/\mathcal{D} \quad (1.1)$$

defined by $\mathcal{L}(v, w) := [X^{(v)}, X^{(w)}]_x \bmod \mathcal{D}|_x$, where $v, w \in \mathcal{D}|_x$ and $X^{(v)}$ and $X^{(w)}$ are sections of \mathcal{D} that extends respectively v and w around $x \in M$. It is the first main invariant of $(M, \mathcal{D}, \mathcal{J})$ and it is called nondegenerate if for any nonzero $v \in \mathcal{D}|_x$ there is an element $w \in \mathcal{D}|_x$ with $\mathcal{L}(v, w) \neq 0$. The equivalence problem of strongly regular Levi nondegenerate CR manifolds of any CR dimension and codimension was solved by N. Tanaka in [31, 32] using a generalization of the usual prolongation procedure of G -structures [29]. He observed that, under fairly general assumptions, any distribution \mathcal{D} on a manifold M determines a filtration in negative degrees of each tangent space (see §2.1 for the definition of this sequence)

$$T_x M = \mathcal{D}_{-\mu}|_x \supset \mathcal{D}_{-\mu+1}|_x \supset \cdots \supset \mathcal{D}_{-2}|_x \supset \mathcal{D}_{-1}|_x = \mathcal{D}|_x, \quad (1.2)$$

and that the associated \mathbb{Z} -graded vector space

$$\mathfrak{m}_x = \text{gr}(T_x M) = \mathfrak{m}_x^{-\mu} \oplus \mathfrak{m}_x^{-\mu+1} \oplus \cdots \oplus \mathfrak{m}_x^{-2} \oplus \mathfrak{m}_x^{-1}$$

inherits, by the commutators of vector fields, a natural structure of \mathbb{Z} -graded Lie algebra $\mathfrak{m}_x = \bigoplus_{p \in \mathbb{Z}} \mathfrak{m}_x^p$ which enjoys the following properties:

- (i) \mathfrak{m}_x is *negatively* \mathbb{Z} -graded, that is $\mathfrak{m}_x^p = 0$ for all $p \geq 0$,
- (ii) \mathfrak{m}_x is of *depth* μ , that is $\mathfrak{m}_x^p = 0$ for all $p < -\mu$,
- (iii) \mathfrak{m}_x is *fundamental*, that is it is generated by \mathfrak{m}_x^{-1} .

He assumed that (M, \mathcal{D}) is strongly regular of type \mathfrak{m} , i.e. with all \mathfrak{m}_x isomorphic to a fixed fundamental \mathbb{Z} -graded Lie algebra $\mathfrak{m} = \mathfrak{m}^{-\mu} \oplus \cdots \oplus \mathfrak{m}^{-1}$, and noted that the presence of an additional geometric datum supported on the distribution \mathcal{D} corresponds to the assignment of a subalgebra \mathfrak{g}^0 of the Lie algebra $\mathfrak{der}(\mathfrak{m})$ of all zero-degree derivations of \mathfrak{m} . In the case of CR manifolds the existence of \mathcal{J} corresponds to the existence of a complex structure $J : \mathfrak{m}^{-1} \longrightarrow \mathfrak{m}^{-1}$ satisfying $[Jv, Jw] = [v, w]$ for all $v, w \in \mathfrak{m}^{-1}$,

$$\mathfrak{g}^0 = \mathfrak{der}(\mathfrak{m}, J) = \{X \in \mathfrak{der}(\mathfrak{m}) \mid X|_{\mathfrak{m}^{-1}} \circ J = J \circ X|_{\mathfrak{m}^{-1}}\}$$

and the CR structure can be encoded in an appropriate principal bundle $\pi : P \longrightarrow M$ of “graded frames” on M with structure group G^0 , $\text{Lie}(G^0) = \mathfrak{g}^0$.

He then showed that any pair $(\mathfrak{m}, \mathfrak{g}^0)$ admits a unique maximal transitive prolongation to positive degrees, a (possibly infinite-dimensional) \mathbb{Z} -graded Lie algebra

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^p \quad (1.3)$$

satisfying:

- (i) \mathfrak{g}^p is finite-dimensional for every $p \in \mathbb{Z}$,
- (ii) $\mathfrak{g}^p = \mathfrak{m}^p$ for every $-\mu \leq p \leq -1$, \mathfrak{g}^0 is equal to the given subalgebra of $\mathfrak{der}(\mathfrak{m})$ and $\mathfrak{g}^p = 0$ for every $p < -\mu$;

- (iii) for all $p \geq 0$, if $X \in \mathfrak{g}^p$ is an element such that $[X, \mathfrak{g}^{-1}] = 0$, then $X = 0$ (*transitivity*);
- (iv) \mathfrak{g} is *maximal* with these properties.

In the case of CR manifolds and $\mathfrak{g}^0 = \mathfrak{der}(\mathfrak{m}, J)$ one has that \mathfrak{g} is of finite type, that is there is a nonnegative integer ℓ such that $\mathfrak{g}^p = 0$ for all $p > \ell$, if and only if for any nonzero $v \in \mathfrak{m}^{-1}$ there is $w \in \mathfrak{m}^{-1}$ with $[v, w] \neq 0$ (see [32]; see also [19, 20] for further properties of (1.3)). This condition clearly corresponds to the nondegeneracy of the Levi form (1.1). In this case the nilpotent Lie group associated with \mathfrak{m} has a natural left-invariant CR structure and is locally identifiable with the homogeneous model $M = G/G_o$, where $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(G_o) = \bigoplus_{p \geq 0} \mathfrak{g}^p$. Moreover there is a finite tower

$$\dots \longrightarrow P^i \xrightarrow{\pi_i} P^{i-1} \longrightarrow \dots \longrightarrow P^1 \xrightarrow{\pi_1} P \xrightarrow{\pi} M$$

of bundles $\pi_i : P^i \longrightarrow P^{i-1}$ with abelian structure groups \mathfrak{g}^i with the property that any CR automorphism of $(M, \mathcal{D}, \mathcal{J})$ admits a unique lift to each term of the tower and thus ending with an automorphism of the absolute parallelism $\pi_{\ell+1} : P^{\ell+1} \longrightarrow P^\ell$ (in some notable cases a Cartan connection modeled on $M = G/G_o$, cf. [33, 34, 1]). This reduces the equivalence problem of strongly regular Levi-nondegenerate CR manifolds to that of the associated absolute parallelisms, which can be dealt with classical results [29].

Note that an important result of the prolongation procedure is given by the fact that both \mathfrak{m} and \mathfrak{g} are \mathbb{Z} -graded Lie algebras, with compatible \mathbb{Z} -gradings. Indeed the idea of considering the filtration (1.2) was also independently proposed by B. Weisfeiler in [36] and the associated \mathbb{Z} -graded Lie algebras of depth $\mu > 1$ were used for filling a gap in E. Cartan classification of the infinite primitive Lie algebras of vector fields.

Unfortunately, for *degenerate* CR structures, the Tanaka approach does not work since the prolongation (1.3) is infinite. In this paper we propose a way to overcome this difficulty. We recall that the first examples of Levi-degenerate CR manifolds are the products $M = \overline{M} \times \mathbb{C}^s$ where \overline{M} is Levi-nondegenerate. More generally any CR manifold $(M, \mathcal{D}, \mathcal{J})$ is endowed also with a second natural filtration. It is an increasing filtration of $\mathcal{D}^{10}|_x$ (see [12] and §2.1)

$$\mathcal{D}^{10}|_x = \mathcal{F}_{-1}^{10}|_x \supset \mathcal{F}_0^{10}|_x \supset \mathcal{F}_1^{10}|_x \supset \dots \supset \mathcal{F}_p^{10}|_x \supset \mathcal{F}_{p+1}^{10}|_x \supset \dots \quad (1.4)$$

which stabilizes at a certain point $k - 1 \geq -1$, i.e. with $\mathcal{F}_{k-1+p}^{10} = \mathcal{F}_{k-1}^{10}$ for all $p \geq 0$, and with the property that the first stabilizing distribution \mathcal{F}_{k-1}^{10} is nonzero if and only if $(M, \mathcal{D}, \mathcal{J})$ is locally a product as above. The CR manifolds $(M, \mathcal{D}, \mathcal{J})$ with $\mathcal{F}_{k-1}^{10} = 0$ are called k -nondegenerate (for $k = 1$ they reduce to the Levi-nondegenerate CR manifolds) and they are not, even locally, products of the form $M = \overline{M} \times \mathbb{C}^s$. Our starting point is very simple and can shortly be summarized as follows: we combine filtrations (1.2) and (1.4) into a single filtration and consider the resulting pointwise invariant \mathfrak{m}_x (called *core* throughout the paper). In a sense the main idea of the paper is complementary to the one of Tanaka and Weisfeiler: the invariant \mathfrak{m}_x not only has a depth $\mu > 1$ but, since the CR manifold (M, \mathcal{D}, J) is k -nondegenerate, it also has a nonnegative height. Indeed $\mathfrak{m}_x^p \neq 0$ exactly if $-\mu \leq p \leq k - 2$, with case $k = 1$ reducing to the case of nondegenerate CR manifolds and negatively graded Lie algebras.

More precisely the **main aim** of this paper is to introduce a notion of *homogeneous model* $M = G/G_o$ of type \mathfrak{m} for k -nondegenerate CR manifolds when $k \geq 2$. This has a double motivation. On one hand it provides with a method to construct CR manifolds with a uniformly degenerate Levi form. We remark that the construction of CR manifolds which are uniformly k -nondegenerate at all points with $k \geq 2$ is a difficult problem and that homogeneous CR manifolds have been constructed in [9, 10, 11, 16, 21], using various techniques. Our definition of a model is new and gives another means to build up k -nondegenerate homogeneous CR

manifolds. We stress that our results are effective and can in principle be applied without any restriction on the dimension m . Moreover, since the construction of models is tightly related to the cores \mathfrak{m} , we also automatically have that two models $M = G/G_o$ and $M' = G'/G'_o$ of different types \mathfrak{m} and \mathfrak{m}' are not, even locally, CR diffeomorphic.

On the other hand the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ of infinitesimal CR automorphisms of $M = G/G_o$ satisfies properties which directly generalize those of the maximal prolongation (1.3) and we expect these homogeneous manifolds as natural candidates for solutions of equivalence problems, especially for the construction of Cartan connections. On this regard, we remark that the equivalence problem for all 5-dimensional 2-nondegenerate CR manifolds had been recently settled in [17, 22]. In particular the Cartan connection “à la Tanaka” determined in [22] is modeled on the projective completion $M = \text{SO}^o(3, 2)/G_o$ of the tube over the future light cone (see [10, 26] for its main properties) and the tower of fiber bundles is constructed as the geometric counterpart of a special \mathbb{Z} -grading of the Lie algebra $\text{Lie}(G) = \mathfrak{so}(3, 2)$, which is indeed of the form we study in this paper (see [22, §3.2] and Example 4.5). The solution of the same equivalence problem presented in [17] uses different methods and the associated set of invariants does not correspond to a Cartan connection modeled on $M = \text{SO}^o(3, 2)/G_o$. The equivalence problem of 7-dimensional 2-nondegenerate CR manifolds has been recently considered in [25] under certain additional constraints. The solution is given by an absolute parallelism taking values in $\mathfrak{g} = \mathfrak{su}(2, 2)$ or $\mathfrak{g} = \mathfrak{su}(1, 3)$, the algebras of infinitesimal CR automorphisms of two of the seven 7-dimensional 2-nondegenerate homogeneous models we obtain in this paper.

We now give the detailed description of the contents of the paper.

In §2.1 we recall the relevant definitions and combine the Tanaka (1.2) and Freeman (1.4) sequences to construct the core $\mathfrak{m}_x = \bigoplus_{p \leq k-2} \mathfrak{m}_x^p$ associated with a k -nondegenerate CR manifold $(M, \mathcal{D}, \mathcal{J})$ at a point $x \in M$ (Definition 2.2 and Lemma 2.3). It is an invariant which generalizes the notion of a fundamental algebra of a nondegenerate CR manifold but in sharp contrast with this case it turns out that the collection of all Levi forms of higher order does not induce a structure of a Lie algebra on \mathfrak{m}_x . Sections §3 and §4.1 are therefore entirely dedicated to constructing homogeneous CR manifolds with a given (abstract) core \mathfrak{m} . To this aim, we first recall in §2.2 the correspondence developed in [21, 11] between germs of homogeneous CR manifolds $M = G/G_o$ and CR algebras $(\mathfrak{g}, \mathfrak{q})$, i.e. Lie algebras of local infinitesimal CR automorphisms, and note that the core can be easily read off from $(\mathfrak{g}, \mathfrak{q})$. In other words we consider homogeneous CR manifolds from a local point of view. In particular in this paper we will not address the question of their global existence in full generality, which would involve criteria for the closedness of the isotropy subgroup, but we will easily check that on a case by case basis.

Starting from §3 we restrict to the case of CR manifolds of hypersurface type, that is with $c = 1$. We recognize higher order Levi forms as defining components of the Tanaka-Weisfeiler grading of the real contact algebra \mathfrak{c} (see [23, 28]) and describe in Proposition 3.4 a CR algebra $(\mathfrak{c}, \mathfrak{u})$ which is *universal* in the sense that any abstract core \mathfrak{m} (of hypersurface type) has a natural immersion $\varphi : \mathfrak{m} \rightarrow \mathfrak{M}$ into the core \mathfrak{M} of $(\mathfrak{c}, \mathfrak{u})$. We remark that the Lie algebra \mathfrak{c} is infinite-dimensional, consistent with the fact that it has to accommodate for abstract cores $\mathfrak{m} = \bigoplus_{p \leq k-2} \mathfrak{m}^p$ of any possible height $\text{ht}(\mathfrak{m}) = k - 2$. In Proposition 3.2 we describe the Lie algebra structure of \mathfrak{c} , generalizing a description of Morimoto and Tanaka [23] (the recursive expression of the brackets is rather involved but it is fully needed in the construction of the 7-dimensional 3-nondegenerate homogeneous model in §6).

In Definition 4.1 we give the definition of a *model of type* \mathfrak{m} as an appropriate \mathbb{Z} -graded subalgebra $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of the universal CR algebra tightly related to \mathfrak{m} and prove in Theorem 4.2 of §4.1 that any model \mathfrak{g} determines a CR algebra $(\mathfrak{g}, \mathfrak{q})$ with core \mathfrak{m} in a canonical way. We note that the real isotropy algebra \mathfrak{g}_o of $(\mathfrak{g}, \mathfrak{q})$ satisfies $\mathfrak{g}/\mathfrak{g}_o \simeq \mathfrak{m}$ and that it is a *proper* subalgebra of the nonnegative part $\bigoplus_{p \geq 0} \mathfrak{g}^p$ of \mathfrak{g} whenever $k \geq 2$. Section 4.1 ends with Example 4.4 and Example 4.5.

In §4.2, §5 and §6 we consider applications of the general theory to the case of CR manifolds of dimension $m = 7$. We remark that this is the smallest possible dimension for a CR hypersurface to be 3-nondegenerate and that a full classification of homogeneous 2-nondegenerate hypersurfaces in dimension $m = 5$ up to local CR equivalence had been given in [10]. In order to construct models which are inequivalent, we first classify in §4.2 the 7-dimensional abstract cores up to isomorphism, see Theorem 4.9 and Theorem 4.11. This result might be of independent interest, as the core is a basic invariant of any 7-dimensional CR manifold. The proof relies on a down-to-earth description of representatives for the orbit spaces of the actions of $SO_3(\mathbb{R})$ and $SO^+(2, 1)$ on the complex projective plane $\mathbb{P}^2(\mathbb{C})$ (see Proposition 4.8 and Proposition 4.10). Finally we introduce a notion of admissibility for an orbit and the associate abstract core in Definition 4.7.

In §5 and §6 we construct models \mathfrak{g} corresponding to the abstract cores \mathfrak{m} determined in §4.2. In Theorem 5.1 of §5 we prove the existence of \mathfrak{g} for all the admissible 2-nondegenerate abstract cores. More precisely we obtain three homogeneous models given by some \mathbb{Z} -gradings $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of the simple Lie algebras $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$, $\mathfrak{su}(1, 3)$ and $\mathfrak{su}(2, 2)$ in Theorem 5.3 and four other models of the form $\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0$ in Theorem 5.4. We finally show in Theorem 6.1 that there exists a unique model in the 3-nondegenerate class.

Before concluding, we recall that a 7-dimensional 2-nondegenerate (resp. 3-nondegenerate) CR manifold locally homogeneous with respect to an algebra \mathfrak{g} isomorphic to $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(1, 3)$ (resp. with $\dim(\mathfrak{g}) = 8$) had also appeared in [16, 25] (resp. [10]). We also remark that a concept of model has been introduced already in [4], in a more analytic context. Analogies and differences with [4] will be discussed elsewhere.

Notations. Given a real vector space V we set $V^\times = \{v \in V \mid v \neq 0\}$ and $V^\mathbb{C} = V \otimes \mathbb{C}$. In the case of models and cores we also use the shortcuts $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ and $\widehat{\mathfrak{m}} = \mathfrak{m} \otimes \mathbb{C}$. We denote by $\underline{\mathcal{K}}$ the space of sections of a bundle \mathcal{K} on M and decompose any section X of $\mathcal{D}^\mathbb{C}$ into the sum $X = X^{10} + X^{01}$ of its holomorphic $X^{10} \in \underline{\mathcal{D}}^{10}$ and antiholomorphic $X^{01} \in \underline{\mathcal{D}}^{01}$ parts.

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2. MAIN DEFINITIONS AND PRELIMINARIES

2.1. The sequences of Tanaka and Freeman and the abstract cores. The *Tanaka sequence* of a CR manifold $(M, \mathcal{D}, \mathcal{J})$ (see [32]) is the nested sequence of $\mathcal{C}^\infty(M)$ -modules of real vector fields

$$\cdots \supset \underline{\mathcal{D}}_{p-1} \supset \underline{\mathcal{D}}_p \supset \underline{\mathcal{D}}_{p+1} \supset \cdots \supset \underline{\mathcal{D}}_{-3} \supset \underline{\mathcal{D}}_{-2} \supset \underline{\mathcal{D}}_{-1}$$

iteratively defined for any $p \leq -1$ by

$$\underline{\mathcal{D}}_{-1} := \underline{\mathcal{D}} \quad \text{and} \quad \underline{\mathcal{D}}_p := \underline{\mathcal{D}}_{p+1} + [\underline{\mathcal{D}}_{-1}, \underline{\mathcal{D}}_{p+1}] .$$

On the other hand, the *Freeman sequence* (see [12]) is a sequence

$$\underline{\mathcal{F}}_{-1} \supset \underline{\mathcal{F}}_0 \supset \underline{\mathcal{F}}_1 \supset \cdots \supset \underline{\mathcal{F}}_{p-1} \supset \underline{\mathcal{F}}_p \supset \underline{\mathcal{F}}_{p+1} \supset \cdots$$

of complex vector fields given for any $p \geq -1$ by

$$\underline{\mathcal{F}}_p = \underline{\mathcal{F}}_p^{10} \oplus \overline{\underline{\mathcal{F}}_p^{10}} \quad \text{where} \quad \underline{\mathcal{F}}_{-1}^{10} := \underline{\mathcal{D}}^{10} \quad \text{and}$$

$$\underline{\mathcal{F}}_p^{10} := \left\{ X \in \underline{\mathcal{F}}_{p-1}^{10} : [X, \underline{\mathcal{D}}^{01}] = 0 \mod \underline{\mathcal{F}}_{p-1}^{10} \oplus \underline{\mathcal{D}}^{01} \right\};$$

note that $\underline{\mathcal{F}}_{-1} = \underline{\mathcal{D}}_{-1}^{\mathbb{C}} = \underline{\mathcal{D}}^{\mathbb{C}}$ whereas $\underline{\mathcal{F}}_p^{10}$ for $p \geq 0$ coincides with the left kernel of the Levi form of higher order

$$\begin{aligned} \mathcal{L}^{p+1} : \underline{\mathcal{F}}_{p-1}^{10} \times \underline{\mathcal{D}}^{01} &\longrightarrow \mathfrak{X}(M)^{\mathbb{C}} / (\underline{\mathcal{F}}_{p-1}^{10} \oplus \underline{\mathcal{D}}^{01}) \\ (X, Y) &\longrightarrow [X, Y] \mod \underline{\mathcal{F}}_{p-1}^{10} \oplus \underline{\mathcal{D}}^{01}. \end{aligned}$$

Each term $\underline{\mathcal{F}}_p$ of the Freeman sequence is a $\mathcal{C}^\infty(M)$ -module and, if $p \geq 0$, also a Lie subalgebra of $\mathfrak{X}(M)^{\mathbb{C}}$. We say that $(M, \mathcal{D}, \mathcal{J})$ is *regular* if the vector fields in $\underline{\mathcal{D}}_p$ and $\underline{\mathcal{F}}_p^{10}$ are the sections of corresponding distributions $\mathcal{D}_p \subset TM$ and $\mathcal{F}_p^{10} \subset \mathcal{D}^{10}$ and if $\mathcal{D}_{-\mu} = TM$ for some positive integer μ . From now on, *any CR manifold is assumed to be regular*.

Definition 2.1. [2, 16] A CR manifold is *k-nondegenerate* if $\mathcal{F}_p \neq 0$ for all $-1 \leq p \leq k-2$ and $\mathcal{F}_{k-1} = 0$.

Definition 2.2. An *abstract core* is a finite dimensional \mathbb{Z} -graded real vector space $\mathfrak{m} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{m}^p$ endowed with

- (i) a complex structure

$$J : \bigoplus_{p \geq -1} \mathfrak{m}^p \longrightarrow \bigoplus_{p \geq -1} \mathfrak{m}^p$$

compatible with the grading, i.e. with $J(\mathfrak{m}^p) \subset \mathfrak{m}^p$ for all $p \geq -1$. We denote by $\widehat{\mathfrak{m}}^p = \mathfrak{m}^{p(10)} \oplus \mathfrak{m}^{p(01)}$ the corresponding decomposition into J -holomorphic and J -antiholomorphic parts of $\widehat{\mathfrak{m}}^p = \mathfrak{m}^p \otimes \mathbb{C}$ for all $p \geq -1$;

- (ii) a bracket of \mathbb{Z} -graded Lie algebras

$$[\cdot, \cdot] : \mathfrak{m}_- \wedge \mathfrak{m}_- \longrightarrow \mathfrak{m}_-, \quad \mathfrak{m}_- = \bigoplus_{p < 0} \mathfrak{m}^p,$$

satisfying $[Jv, Jw] = [v, w]$ for all $v, w \in \mathfrak{m}^{-1}$ and nondegenerate and fundamental in Tanaka's sense;

- (iii) an injective and \mathbb{C} -linear map for any $p \geq 0$

$$L^{p+2} : \mathfrak{m}^{p(10)} \longrightarrow \mathfrak{m}^{p-1(10)} \otimes (\mathfrak{m}^{-1(01)})^* \bigcap \widehat{\mathfrak{m}}^{-2} \otimes S^{p+2}(\mathfrak{m}^{-1(01)})^*$$

where $\mathfrak{m}^{-1(10)}$ is understood as space of maps from $\mathfrak{m}^{-1(01)}$ to $\widehat{\mathfrak{m}}^{-2}$ using bracket (ii).

The core is of *depth* $d(\mathfrak{m}) = \mu$ (resp. *height* $ht(\mathfrak{m}) = k-2$) if $\mathfrak{m}^p = 0$ for all $p < -\mu$ (resp. all $p > k-2$). A *morphism* of cores \mathfrak{m} and \mathfrak{m}' is a morphism $\varphi : \mathfrak{m} \longrightarrow \mathfrak{m}'$ of real vector spaces such that

- (i) $\varphi(\mathfrak{m}^p) \subset \mathfrak{m}'^p$ for all $p \in \mathbb{Z}$;
- (ii) $\varphi|_{\mathfrak{m}^p} \circ J = J' \circ \varphi|_{\mathfrak{m}^p}$ for all $p \geq -1$;
- (iii) $\varphi|_{\mathfrak{m}_-} : \mathfrak{m}_- \longrightarrow \mathfrak{m}'_-$ is a morphism of Lie algebras;

- (iv) $\varphi^*(L'^{p+2} \circ \varphi(v)) = \varphi \circ L^{p+2}(v)$ for all $v \in \mathfrak{m}^{p(10)}$, $p \geq 0$, where φ is extended by \mathbb{C} -linearity.

It is an *immersion* (resp. an *isomorphism*) if it is injective (resp. bijective).

Note that any immersion φ with $\varphi(\mathfrak{m}^{-1}) = \mathfrak{m}'^{-1}$ is fully determined by its action on \mathfrak{m}_- , by property (iv). The following result recasts the Tanaka and Freeman sequences, and the higher order Levi forms, in the form suitable for our purposes.

Lemma 2.3. *Let $(M, \mathcal{D}, \mathcal{J})$ be a k -nondegenerate CR manifold with $\mathcal{D}_{-\mu} = TM$. For every $x \in M$, the \mathbb{Z} -graded vector space $\mathfrak{m}_x = \bigoplus_{p \in \mathbb{Z}} \mathfrak{m}_x^p$ defined by*

$$\begin{aligned} \mathfrak{m}_x^p &= \frac{\mathcal{D}_p|_x}{\mathcal{D}_{p+1}|_x} & \text{for all } p \leq -2, \\ \mathfrak{m}_x^p &= \frac{\Re(\mathcal{F}_p)|_x}{\Re(\mathcal{F}_{p+1})|_x} & \text{for all } p \geq -1, \end{aligned}$$

has the natural structure of an abstract core of depth μ and height $k - 2$.

Proof. The depth and height follow directly from definitions while point (i) of Definition 2.2 from the fact that \mathcal{J} preserves the real part $\Re(\mathcal{F}_p)$ of \mathcal{F}_p for every $p \geq -1$. The proof of (ii) follows the same lines of [32, §1.2] once we had observed $[\mathcal{F}_0, \mathcal{D}_p] \subset \mathcal{D}_p$ for every $p \leq -1$. Now for every $p \geq 0$ we have $[\mathcal{F}_p^{10}, \mathcal{D}^{01}] \subset \mathcal{F}_{p-1}^{10} \oplus \mathcal{D}^{01}$ and $[\mathcal{F}_p^{10}, \mathcal{F}_0^{01}] \subset \mathcal{F}_p^{10} \oplus \mathcal{F}_0^{01}$ (see [16, Appendix]) and the higher order Levi form induces a well-defined bilinear map

$$\mathcal{L}_x^{p+2} : \mathfrak{m}_x^{p(10)} \times \mathfrak{m}_x^{-1(01)} \longrightarrow \frac{\mathcal{F}_{p-1}^{10}|_x \oplus \mathcal{D}^{01}|_x}{\mathcal{F}_p^{10}|_x \oplus \mathcal{D}^{01}|_x} \simeq \mathfrak{m}_x^{p-1(10)}$$

at any $x \in M$. The corresponding map from $\mathfrak{m}_x^{p(10)}$ to $\mathfrak{m}_x^{p-1(10)} \otimes (\mathfrak{m}_x^{-1(01)})^*$ is injective and \mathbb{C} -linear and, by a direct induction on $p \geq 0$, takes values in the subspace $\widehat{\mathfrak{m}}_x^{-2} \otimes S^{p+2}(\mathfrak{m}_x^{-1(01)})^*$ of $\mathfrak{m}_x^{-1(10)} \bigotimes^{p+1} (\mathfrak{m}_x^{-1(01)})^*$. \square

By Definition 2.2 and Lemma 2.3, the core is an invariant of a CR manifold which generalizes the usual notion of a fundamental algebra associated with a nondegenerate CR manifold ([32]). In contrast with this case, it does not possess any structure of a Lie algebra and the problem of constructing k -nondegenerate CR manifolds with a given core at all points is more delicate. To do so, we first need to recall the notions of homogeneous CR manifolds and their associated CR algebras.

2.2. Homogeneous CR manifolds and CR algebras. Let $(M, \mathcal{D}, \mathcal{J})$ be a CR manifold that is locally homogeneous around $x \in M$ under a (possibly infinite-dimensional) Lie algebra \mathfrak{g} of infinitesimal CR automorphisms. Transitivity of the action amounts to $T_x M = \{Z|_x \mid Z \in \mathfrak{g}\}$ while

$$\mathfrak{q} = \left\{ Z \in \widehat{\mathfrak{g}} \mid Z|_x \in \mathcal{D}^{10}|_x \right\}$$

is a complex Lie subalgebra of $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$, by the integrability condition of CR manifolds. In the terminology of [21] the pair $(\mathfrak{g}, \mathfrak{q})$ is an *abstract CR algebra*, i.e. it enjoys:

- (i) \mathfrak{g} is a real Lie algebra,
- (ii) \mathfrak{q} is a complex subalgebra of $\widehat{\mathfrak{g}}$,
- (iii) the quotient $\mathfrak{g}/\mathfrak{g}_o$ is finite-dimensional, where

$$\mathfrak{g}_o = \mathfrak{g} \cap \mathfrak{q} = \Re(\mathfrak{q} \cap \bar{\mathfrak{q}}) = \left\{ Z \in \mathfrak{g} \mid Z|_x = 0 \right\}$$

is the real isotropy algebra at x .

Conversely any abstract CR algebra determines a unique (germ of) locally homogeneous CR manifold $(M, \mathcal{D}, \mathcal{J})$ with

$$\begin{aligned} T_x M &\simeq \mathfrak{g}/\mathfrak{g}_o, \\ \mathcal{D}^{10}|_x &\simeq \mathfrak{q}/\mathfrak{q} \cap \bar{\mathfrak{q}}, \end{aligned} \quad (2.1)$$

see [21] and also [11, §4] for more details. The associated core at $x \in M$ can also be directly recovered from the CR algebra. First note that the Freeman bundles are locally homogeneous bundles with fiber $\mathcal{F}_p^{10}|_x \simeq \mathfrak{q}_p/\mathfrak{q} \cap \bar{\mathfrak{q}}$ where

$$\begin{aligned} \mathfrak{q}_p &= \left\{ Z \in \widehat{\mathfrak{g}} \mid Z|_x \in \mathcal{F}_p^{10}|_x \right\} \\ &= \left\{ Z \in \mathfrak{q}_{p-1} \mid [Z, \bar{\mathfrak{q}}] \subset \mathfrak{q}_{p-1} + \bar{\mathfrak{q}} \right\} \end{aligned} \quad (2.2)$$

and $\mathfrak{q}_{-1} = \mathfrak{q} \supset \mathfrak{q}_0 \supset \cdots \supset \mathfrak{q}_{p-1} \supset \mathfrak{q}_p \supset \mathfrak{q}_{p+1} \supset \cdots \supset \mathfrak{q} \cap \bar{\mathfrak{q}}$ is the associated sequence of complex subalgebras of \mathfrak{q} (see [11]). The homogeneous CR manifold is k -nondegenerate precisely when $\mathfrak{q}_{k-1} = \mathfrak{q} \cap \bar{\mathfrak{q}}$ and $\mathfrak{q}_{k-2} \neq \mathfrak{q} \cap \bar{\mathfrak{q}}$. Similarly the fibers $\mathcal{D}_p|_x \simeq \mathfrak{g}_p/\mathfrak{g}_o$ of the Tanaka bundles correspond to the sequence $\mathfrak{g}_{-\mu} \supset \cdots \supset \mathfrak{g}_{p-1} \supset \mathfrak{g}_p \supset \mathfrak{g}_{p+1} \supset \cdots \supset \mathfrak{g}_{-2} \supset \mathfrak{g}_{-1}$ of subspaces

$$\mathfrak{g}_{-1} = \Re(\mathfrak{q} + \bar{\mathfrak{q}}), \quad \mathfrak{g}_p = \mathfrak{g}_{p+1} + [\mathfrak{g}_{-1}, \mathfrak{g}_{p+1}]. \quad (2.3)$$

It is not difficult to see that $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for all $p, q \leq -1$. It follows from these observations that

$$\mathfrak{m}_x^p \simeq \mathfrak{g}_p/\mathfrak{g}_{p+1} \quad \text{for all } p \leq -2, \quad (2.4)$$

$$\mathfrak{m}_x^p \simeq \Re\left(\frac{\mathfrak{q}_p + \bar{\mathfrak{q}}_p}{\mathfrak{q}_{p+1} + \bar{\mathfrak{q}}_{p+1}}\right) \quad \text{for all } p \geq -1, \quad (2.5)$$

$$\mathfrak{m}_x^{p(10)} \simeq \mathfrak{q}_p/\mathfrak{q}_{p+1}$$

as (real or complex) vector spaces and the Lie bracket of \mathfrak{m}_- is induced by the Lie bracket of \mathfrak{g} . Finally $[\mathfrak{q}_p, \bar{\mathfrak{q}}_0] \subset \mathfrak{q}_p + \bar{\mathfrak{q}}_0$ for all $p \geq 0$ by a direct induction and the Lie bracket of \mathfrak{g} also induces an operation on the quotients

$$[\mathfrak{m}_x^{p(10)}, \mathfrak{m}_x^{-1(01)}] \subset \frac{\mathfrak{q}_{p-1} + \bar{\mathfrak{q}}}{\mathfrak{q}_p + \bar{\mathfrak{q}}} \simeq \mathfrak{m}_x^{p-1(10)}$$

and, in turn, the required immersion L^{p+2} of (iii) of Definition 2.2.

In view of §3 it is convenient to relax the definition of a CR algebra, including those pairs which satisfy (i) and (ii), but not necessarily (iii). Informally speaking we are considering infinite-dimensional locally homogeneous CR manifolds but we will not attempt to rigorously define such objects. Note however that the r.h.s. of (2.1) and (2.2)-(2.5) still make sense and we may say that $(\mathfrak{g}, \mathfrak{q})$ is holomorphically nondegenerate when $\bigcap_{p \geq -1} \mathfrak{q}_p = \mathfrak{q} \cap \bar{\mathfrak{q}}$ (this corresponds to usual k -nondegeneracy of some order k whenever $\dim(\mathfrak{g}/\mathfrak{g}_o) < +\infty$).

In §3 and §4 we consider the opposite problem of associating a CR algebra to an abstract core. We restrict to CR manifolds $(M, \mathcal{D}, \mathcal{J})$ of hypersurface type and therefore to cores \mathfrak{m} of depth $d(\mathfrak{m}) = 2$ and $\dim(\mathfrak{m}^{-2}) = 1$. We first describe in §3 an holomorphically nondegenerate CR algebra $(\mathfrak{c}, \mathfrak{u})$ that is universal, in the sense that any abstract core \mathfrak{m} has a natural immersion $\varphi : \mathfrak{m} \rightarrow \mathfrak{M}$ into the core \mathfrak{M} of $(\mathfrak{c}, \mathfrak{u})$, see Proposition 3.4. This universal property will be crucial to constructing appropriate Lie subalgebras of $(\mathfrak{c}, \mathfrak{u})$ and their associated homogeneous CR manifolds.

3. THE UNIVERSAL CR ALGEBRA

The universal CR algebra $(\mathfrak{c}, \mathfrak{u})$ is given by the real infinite-dimensional contact algebra \mathfrak{c} and an appropriate complex subalgebra \mathfrak{u} of its complexification; we first study the structure of contact algebras over $\mathbb{K} = \mathbb{C}, \mathbb{R}$ simultaneously. Let $\mathfrak{c}_- = \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1}$ be the Heisenberg algebra of degree n , the fundamental \mathbb{Z} -graded Lie algebra over \mathbb{K} such that:

- (i) $\dim \mathfrak{c}^{-2} = 1$,
- (ii) $\dim \mathfrak{c}^{-1} = 2n$,
- (iii) \mathfrak{c}_- is nondegenerate (if $v \in \mathfrak{c}^{-1}$ satisfies $[v, \mathfrak{c}^{-1}] = 0$ then $v = 0$).

It is well-known that the maximal transitive prolongation of \mathfrak{c}_- is an infinite-dimensional simple Lie algebra with the grading

$$\mathfrak{c} = \bigoplus_{p \geq -2} \mathfrak{c}^p, \quad (3.1)$$

usually referred to as the *contact algebra* of degree n [23]. We identify \mathfrak{c}^{-2} with the ground field using the isomorphism $\pi_{\mathfrak{c}^{-2}} : \mathfrak{c}^{-2} \rightarrow \mathbb{K}$ associated with a basis $\{e^{-2}\}$ of \mathfrak{c}^{-2} and denote by $B : \mathfrak{c}^{-1} \wedge \mathfrak{c}^{-1} \rightarrow \mathbb{K}$ the symplectic form

$$[v, w] = B(v, w)e^{-2}, \quad v, w \in \mathfrak{c}^{-1}. \quad (3.2)$$

For all $p \geq -2$, we denote the subspace of \mathfrak{c}^p which acts trivially on \mathfrak{c}^{-2} by

$$\mathfrak{k}^p = \{X \in \mathfrak{c}^p \mid [X, \mathfrak{c}^{-2}] = 0\}$$

and note that $\mathfrak{k}^{-2} = \mathfrak{c}^{-2}$, $\mathfrak{k}^{-1} = \mathfrak{c}^{-1}$ and

$$[\mathfrak{k}^p, \mathfrak{c}^{-1}] \subset \mathfrak{k}^{p-1}, \quad [\mathfrak{k}^p, \mathfrak{k}^q] \subset \mathfrak{k}^{p+q},$$

for all $p, q \geq -1$, since (3.1) is a \mathbb{Z} -graded Lie algebra.

Let E be the element of \mathfrak{c}^0 which satisfies $[E, X] = pX$ for every $X \in \mathfrak{c}^p$, it is called the *grading element*. The 0-degree part of the contact algebra has a direct sum decomposition $\mathfrak{c}^0 = \mathfrak{k}^0 \oplus \mathbb{K}E$, with $\mathfrak{k}^0 = \mathfrak{sp}(\mathfrak{c}^{-1}) = \mathfrak{sp}(\mathfrak{c}^{-1}, B)$. We consider the usual identification $\mathfrak{sp}(\mathfrak{c}^{-1}) \simeq S^2(\mathfrak{c}^{-1})$ where the bracket of an element $v_1 \odot v_2 \in \mathfrak{k}^0$ with $w \in \mathfrak{c}^{-1}$ takes the form

$$[v_1 \odot v_2, w] = v_1 B(v_2, w) + v_2 B(v_1, w). \quad (3.3)$$

It is proved in [23, 28] that there exist similar identifications $\mathfrak{k}^p \simeq S^{p+2}(\mathfrak{c}^{-1})$ which are \mathfrak{k}^0 -equivariant and such that the bracket of $X = v_1 \odot \cdots \odot v_{p+2} \in \mathfrak{k}^p$ with $Y = w_1 \odot \cdots \odot w_{q+2} \in \mathfrak{k}^q$ is

$$[X, Y] = \sum_{i=1}^{p+2} \sum_{j=1}^{q+2} v_1 \odot \cdots \odot \widehat{v}_i \odot \cdots \odot v_{p+2} \odot w_1 \odot \cdots \odot \widehat{w}_j \odot \cdots \odot w_{q+2} B(v_i, w_j),$$

where \widehat{v} indicates that the vector $v \in \mathfrak{c}^{-1}$ is omitted. The following result was also proved by Morimoto and Tanaka.

Proposition 3.1. [23] *Let e^{-2} be a basis of \mathfrak{c}^{-2} . Then $\text{ad}(e^{-2})$ is a surjective and \mathfrak{k}^0 -equivariant map from \mathfrak{c}^p to \mathfrak{c}^{p-2} with kernel \mathfrak{k}^p , for every $p \geq 0$. Moreover there exists a unique \mathfrak{k}^0 -module ξ^p which is isomorphic to \mathfrak{c}^{p-2} and complementary to \mathfrak{k}^p in \mathfrak{c}^p .*

Their proof relied on the existence of appropriate, but not necessarily \mathfrak{k}^0 -equivariant, immersions $\mu^p : \mathfrak{c}^{p-2} \rightarrow \mathfrak{c}^p$. We now give an improved version of this result and an explicit description of the decomposition $\mathfrak{c}^p = \mathfrak{k}^p \oplus \xi^p$. More precisely we give maps μ^p which are \mathfrak{k}^0 -equivariant, together with conditions which guarantee their unicity, and explicitly describe

the Lie brackets between the different irreducible \mathfrak{k}^0 -submodules of \mathfrak{c} (this description will be relevant only in §6 and it can be skipped on a first reading). To formulate the result, we set

$$\begin{aligned} \mu^{-1} &= 0, \quad \mu^{p|p+2} = \text{Id}_{\mathfrak{k}^p} \quad \text{and, for any } 0 \leq i \leq [p/2], \\ \mu^{p|p-2i} &= \mu^p \circ \mu^{p-2} \circ \dots \circ \mu^{p-2i}|_{\mathfrak{k}^{p-2i-2}} : \mathfrak{k}^{p-2i-2} \longrightarrow \mathfrak{c}^p. \end{aligned}$$

We adopt the convention that the binomial coefficient $\binom{-1}{0} = 1$ while $\binom{k-1}{k}$ is trivial for any integer $k \geq 1$.

Proposition 3.2. *There exists a \mathfrak{k}^0 -equivariant immersion $\mu^p : \mathfrak{c}^{p-2} \longrightarrow \mathfrak{c}^p$ with image $\xi^p = \text{Im}(\mu^p)$ and a decomposition $\mathfrak{c}^p = \mathfrak{k}^p \oplus \xi^p$ of \mathfrak{k}^0 -modules for all $p \geq 0$. If $\pi_{\mathfrak{k}^p} : \mathfrak{c}^p \longrightarrow \mathfrak{k}^p$ and $\pi_{\xi^p} : \mathfrak{c}^p \longrightarrow \xi^p$ are the corresponding projection operators, there is a unique choice of μ^p 's such that the following conditions are satisfied*

$$\begin{cases} \mu^p(X)e^{-2} = X & \text{(C1)} \\ \mu^p(X)v = \mu^{p-1}[X, v] + \frac{1}{2}\pi_{\mathfrak{k}^{p-2}}(X) \odot v \in \xi^{p-1} \oplus \mathfrak{k}^{p-1} \text{ for all } v \in \mathfrak{c}^{-1} & \text{(C2)} \end{cases}$$

for all $X \in \mathfrak{c}^{p-2}$. In this case each map $\mu^{p|p-2i}$ is a \mathfrak{k}^0 -equivariant immersion and \mathfrak{c}^p decomposes into irreducible inequivalent \mathfrak{k}^0 -modules as follows:

$$\begin{aligned} \mathfrak{c}^p &\simeq \bigoplus_{-1 \leq i \leq [p/2]} S^{p-2i}(\mathfrak{c}^{-1}), \quad S^{p-2i}(\mathfrak{c}^{-1}) \simeq \text{Im}(\mu^{p|p-2i}) \quad \text{with} \\ \mathfrak{k}^p &\simeq S^{p+2}(\mathfrak{c}^{-1}) \quad \text{and} \quad \xi^p \simeq \bigoplus_{0 \leq i \leq [p/2]} S^{p-2i}(\mathfrak{c}^{-1}). \end{aligned} \quad (3.4)$$

Furthermore:

(i) The bracket of $X \in \mathfrak{k}^p$ with $\mu^{q|q-2j}(Y) \in \xi^q$ is in \mathfrak{c}^{p+q} and

$$[X, \mu^{q|q-2j}(Y)] = \underbrace{\frac{p}{2}\mu^{p+q|p+q+2-2j}(X \odot Y)}_{\text{elem. of } S^{p+q+2-2j}(\mathfrak{c}^{-1})} + \underbrace{\mu^{p+q|p+q-2j}[X, Y]}_{\text{elem. of } S^{p+q-2j}(\mathfrak{c}^{-1})} \quad (3.5)$$

where $p \geq -1$, $q \geq 0$ and $0 \leq j \leq [q/2]$.

(ii) The bracket of $\mu^{p|p-2i}(X) \in \xi^p$ and $\mu^{q|q-2j}(Y) \in \xi^q$ is in ξ^{p+q} and

$$\begin{aligned} [\mu^{p|p-2i}(X), \mu^{q|q-2j}(Y)] &= \alpha(p, i; q, j) \underbrace{\mu^{p+q|p+q-2i-2j}(X \odot Y)}_{\text{elem. of } S^{p+q-2i-2j}(\mathfrak{c}^{-1})} \\ &\quad + \beta(i; j) \underbrace{\mu^{p+q|p+q-2i-2j-2}[X, Y]}_{\text{elem. of } S^{p+q-2i-2j-2}(\mathfrak{c}^{-1})}, \end{aligned} \quad (3.6)$$

where $p, q \geq 0$, $0 \leq i \leq [p/2]$, $0 \leq j \leq [q/2]$ and the coefficients α and β are given respectively by

$$\alpha(p, i; q, j) = \frac{p-2i-2}{2} \left\{ \sum_{k=0}^j \binom{i+k-1}{k} (j+1-k) \right\}$$

$$- \frac{q-2j-2}{2} \left\{ \sum_{k=0}^i \binom{j+k-1}{k} (i+1-k) \right\}$$

and

$$\beta(i; j) = \sum_{k=0}^j \binom{i+k-1}{k} (j+1-k) + \sum_{k=0}^i \binom{j+k-1}{k} (i+1-k) .$$

Proof. It is a straightforward but rather long modification of original arguments of Morimoto and Tanaka in [23] combined with repeated induction arguments. The details are omitted for the sake of brevity. \square

From now on we always denote by (3.4) the decomposition of \mathfrak{c} determined by the \mathfrak{k}^0 -equivariant maps $\mu^{p|p-2i}$ described in Proposition 3.2. Unless there is a risk of confusion, we omit the immersion $\mu^{p|p-2i}$ and tacitly identify its image $\text{Im}(\mu^{p|p-2i})$ in \mathfrak{c}^p with $S^{p-2i}(\mathfrak{c}^{-1})$.

Let now \mathfrak{c} be the *real* contact algebra and $\hat{\mathfrak{c}} = \mathfrak{c} \otimes \mathbb{C}$. We say that a complex structure \mathfrak{J} on \mathfrak{c}^{-1} has signature $\text{sgn}(\mathfrak{J}) = (r, s)$, $n = r + s$, if $\mathfrak{J} \in \mathfrak{k}^0 = \mathfrak{sp}(\mathfrak{c}^{-1})$ and the associated Hermitian product has signature (r, s) . We fix \mathfrak{J} once and for all and identify the space of complex structures of signature (r, s) with $\text{Sp}_{2n}(\mathbb{R})/\text{U}(r, s)$. Clearly $\hat{\mathfrak{c}}^{-1}$ decomposes into $\hat{\mathfrak{c}}^{-1} = \mathfrak{c}^{-1(10)} \oplus \mathfrak{c}^{-1(01)}$ with

$$\begin{aligned} \mathfrak{c}^{-1(10)} &= \left\{ v - i\mathfrak{J}v \mid v \in \mathfrak{c}^{-1} \right\} , \\ \mathfrak{c}^{-1(01)} &= \overline{\mathfrak{c}^{-1(10)}} = \left\{ v + i\mathfrak{J}v \mid v \in \mathfrak{c}^{-1} \right\} . \end{aligned}$$

Similarly we have the following.

Definition 3.3. We indicate by $S^{\ell, p-2j-\ell} = S^{\ell}(\mathfrak{c}^{-1(10)}) \otimes S^{p-2j-\ell}(\mathfrak{c}^{-1(01)})$ the $\text{ad}(\mathfrak{J})$ -eigenspace in $S^{p-2j}(\hat{\mathfrak{c}}^{-1})$ given by $\text{ad}(\mathfrak{J})|_{S^{\ell, p-2j-\ell}} = i(2\ell - p + 2j) \text{Id}$.

It follows that each component $S^{p-2j}(\hat{\mathfrak{c}}^{-1})$ of $\hat{\mathfrak{c}}^p$ decomposes into

$$S^{p-2j}(\hat{\mathfrak{c}}^{-1}) = \bigoplus_{0 \leq \ell \leq p-2j} S^{\ell, p-2j-\ell} \quad (3.7)$$

and there is also a decomposition of the whole $\hat{\mathfrak{c}}$ into $\text{ad}(\mathfrak{J})$ -eigenspaces, with purely imaginary eigenvalues. We now describe the universal CR algebra $(\mathfrak{c}, \mathfrak{u})$.

Proposition 3.4. Let $\mathfrak{u} = \bigoplus_{p \geq -1} \mathfrak{u}^p$ be the \mathbb{Z} -graded subspace of $\hat{\mathfrak{c}}$ where each component

$$\mathfrak{u}^p = \bigoplus_{1 \leq \ell \leq p+2} S^{\ell, p+2-\ell} \oplus \bigoplus_{0 \leq j \leq [p/2]} S^{p-2j}(\hat{\mathfrak{c}}^{-1}) \quad (3.8)$$

is the direct sum of all the $\text{ad}(\mathfrak{J})$ -eigenspaces in $\hat{\mathfrak{c}}^p$ of non-minimal eigenvalue $-i(p+2)$. Then \mathfrak{u} is a Lie subalgebra of $\hat{\mathfrak{c}}$ and $(\mathfrak{c}, \mathfrak{u})$ an holomorphically nondegenerate CR algebra. Moreover:

- (i) the real isotropy algebra is nonnegatively \mathbb{Z} -graded $\mathfrak{c}_o = \bigoplus_{p \geq 0} \mathfrak{c}_o^p$ and each component

$$\begin{aligned} \mathfrak{c}_o^p &= \Re(\mathfrak{u}^p \cap \bar{\mathfrak{u}}^p) \quad \text{where} \\ \mathfrak{u}^p \cap \bar{\mathfrak{u}}^p &= \bigoplus_{1 \leq \ell \leq p+1} S^{\ell, p+2-\ell} \oplus \bigoplus_{0 \leq j \leq [p/2]} S^{p-2j}(\hat{\mathfrak{c}}^{-1}) \end{aligned}$$

is given by the direct sum of all the $\text{ad}(\mathfrak{J})$ -eigenspaces in $\hat{\mathfrak{c}}^p$ of non-extremal eigenvalues $\pm i(p+2)$;

(ii) the components of the core $\mathfrak{M} = \bigoplus_{p \geq -2} \mathfrak{M}^p$ of $(\mathfrak{c}, \mathfrak{u})$ are given by

$$\mathfrak{M}^p = \begin{cases} \mathfrak{c}^p & \text{for } p = -2, -1, \\ \Re(\mathfrak{M}^{p(10)} \oplus \mathfrak{M}^{p(01)}) & \text{for all } p \geq 0, \end{cases}$$

where $\mathfrak{M}^{p(10)} = S^{p+2,0}$ and $\mathfrak{M}^{p(01)} = \overline{\mathfrak{M}^{p(10)}} = S^{0,p+2}$ are the $\text{ad}(\mathfrak{J})$ -eigenspaces in $\hat{\mathfrak{c}}^p$ of extremal eigenvalues $\pm i(p+2)$;

(iii) any abstract core \mathfrak{m} of hypersurface type admits a natural immersion $\varphi : \mathfrak{m} \rightarrow \mathfrak{M}$.

Proof. By definition (3.8), decomposition (3.7) and \mathfrak{k}^0 -equivariance, it follows $[\mathfrak{u}^p, \mathfrak{u}^q] \subset \mathfrak{u}^{p+q}$ and $(\mathfrak{c}, \mathfrak{u})$ is a CR algebra. The claim on \mathfrak{c}_o is also clear as $\mathfrak{c}_o = \Re(\mathfrak{u} \cap \bar{\mathfrak{u}})$ by definition, conjugation in $\hat{\mathfrak{c}}$ with respect to the real form \mathfrak{c} is compatible with the grading and $\mathfrak{u}^{-1} \cap \bar{\mathfrak{u}}^{-1} = 0$. Now a direct induction shows that the p -th term (2.2) of the Freeman sequence of $(\mathfrak{c}, \mathfrak{u})$ is given by

$$\mathfrak{u}_p = \bigoplus_{0 \leq j \leq p-1} (\mathfrak{u}^j \cap \bar{\mathfrak{u}}^j) \oplus \bigoplus_{j \geq p} \mathfrak{u}^j \quad (3.9)$$

so that $\bigcap_{p \geq -1} \mathfrak{u}_p = \mathfrak{u} \cap \bar{\mathfrak{u}}$ and $(\mathfrak{c}, \mathfrak{u})$ is holomorphically nondegenerate.

Claim (ii) follows from (3.9), the decomposition $\hat{\mathfrak{c}} = (\mathfrak{u} \cap \bar{\mathfrak{u}}) \oplus \widehat{\mathfrak{M}}$, the fact that $\mathfrak{M}^{p(10)}$ can be iteratively characterized by $[\mathfrak{M}^{p(10)}, \mathfrak{c}^{-1(01)}] \subset \mathfrak{M}^{p-1(10)}$ for any $p \geq 0$ and, finally, the identifications (2.4)-(2.5). We also note that in this case the immersion

$$\mathfrak{L}^{p+2} : \mathfrak{M}^{p(10)} \rightarrow \mathfrak{M}^{p-1(10)} \otimes (\mathfrak{M}^{-1(01)})^* \cap \widehat{\mathfrak{M}}^{-2} \otimes S^{p+2}(\mathfrak{M}^{-1(01)})^*$$

of (iii) of Definition 2.2 is surjective and hence an isomorphism, for all $p \geq 0$. To prove (iii) fix an abstract core $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ and an identification $\varphi_- : \mathfrak{m}_- \rightarrow \mathfrak{c}_-$ of \mathbb{Z} -graded Lie algebras with $\varphi_-|_{\mathfrak{m}^{-1}} \circ J = \mathfrak{J} \circ \varphi_-|_{\mathfrak{m}^{-1}}$. By the observation after Definition 2.2 and an induction argument we get a unique morphism of cores $\varphi : \mathfrak{m} \rightarrow \mathfrak{M}$ that is injective and satisfies

$$\begin{aligned} \varphi|_{\mathfrak{m}_-} &:= \varphi_-, & \varphi|_{\mathfrak{m}^{p(10)}} &:= (\mathfrak{L}^{p+2})^{-1} \circ \varphi|_{\mathfrak{m}_{<p}} \circ L^{p+2} & \text{and} \\ & & \varphi|_{\mathfrak{m}^{p-1(10)}} \circ J &= \left(\frac{1}{p+1}\mathfrak{J}\right) \circ \varphi|_{\mathfrak{m}^{p-1(10)}}, \end{aligned}$$

where $\mathfrak{m}_{<p} = \bigoplus_{r < p} \mathfrak{m}^r$ for all $p \geq 0$. This concludes the proof. \square

Now any abstract core $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ is, up to isomorphism, always realized as a subspace of \mathfrak{M} with components

$$\mathfrak{m}^p = \begin{cases} 0 & \text{for all } p < -2, \\ \mathfrak{c}^p & \text{for } p = -2, -1, \\ \Re(\mathfrak{m}^{p(10)} \oplus \overline{\mathfrak{m}^{p(10)}}) & \text{for all } p \geq 0, \end{cases} \quad (3.10)$$

where $\mathfrak{m}^{p(10)} \subset \mathfrak{M}^{p(10)}$ satisfies $[\mathfrak{m}^{p(10)}, \mathfrak{c}^{-1(01)}] \subset \mathfrak{m}^{p-1(10)}$ for all $p \geq 1$. In particular

$$\dim_{\mathbb{C}} \mathfrak{m}^{p(10)} \leq \binom{n+p+1}{p+2} \quad (3.11)$$

and two cores $\mathfrak{m}, \mathfrak{m}'$ of the same signature $\text{sgn}(\mathfrak{J}) = (r, s)$ are isomorphic if and only if $\mathfrak{m}' = g \cdot \mathfrak{m}$ for some element $g \in \text{Aut}(\mathfrak{c}, \mathfrak{J})$ of the group

$$\text{Aut}(\mathfrak{c}, \mathfrak{J}) = \left\{ g : \mathfrak{c} \longrightarrow \mathfrak{c} \text{ automorphism of } \mathbb{Z}\text{-graded Lie algebras} \right. \\ \left. \text{such that } g \circ \text{ad}(\mathfrak{J}) = \text{ad}(\mathfrak{J}) \circ g \right\} \simeq \text{CU}(r, s).$$

We have used here that the Lie algebra $\text{cu}(r, s) = \mathbb{R} \oplus \mathfrak{u}(r, s)$ of $\text{Aut}(\mathfrak{c}, \mathfrak{J})$ is naturally identifiable with the subalgebra $\mathbb{R}E \oplus \mathfrak{Re}(S^{1,1})$ of \mathfrak{c}^0 and that any 0-degree automorphism of a fundamental Lie algebra \mathfrak{c}_- can be canonically prolonged to a unique automorphism of the maximal transitive prolongation \mathfrak{c} .

Definition 3.5. A CR manifold $(M, \mathcal{D}, \mathcal{J})$ is called *strongly regular of type \mathfrak{m}* if the associated cores $\mathfrak{m}_x \simeq \mathfrak{m}$ at all points $x \in M$.

The main aim of §4 is to constructing (strongly regular) homogeneous CR manifolds of a given type \mathfrak{m} .

4. HOMOGENEOUS MODELS FOR k -NONDEGENERATE CR MANIFOLDS

4.1. Main definitions, results and first examples. We recall that any abstract core $\mathfrak{m} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{m}^p$ is identified with a finite dimensional subspace (3.10) of the universal CR algebra $(\mathfrak{c}, \mathfrak{u})$.

Definition 4.1. A *model* of type \mathfrak{m} is the datum of a \mathbb{Z} -graded Lie subalgebra

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^p$$

of \mathfrak{c} satisfying

- (i) $\mathfrak{g}^p = \mathfrak{c}^p$ for all $p < 0$;
- (ii) the grading element E belongs to \mathfrak{g}^0 ;
- (iii) $\widehat{\mathfrak{g}}^p = (\widehat{\mathfrak{g}}^p \cap \mathfrak{u}^p) + (\widehat{\mathfrak{g}}^p \cap \bar{\mathfrak{u}}^p)$ for all $p \geq 0$;
- (iv) the natural $\text{Aut}(\mathfrak{c}, \mathfrak{J})$ -equivariant projection $\pi_{\mathfrak{M}^{p(10)}} : \widehat{\mathfrak{c}}^p \longrightarrow \mathfrak{M}^{p(10)}$ of $\widehat{\mathfrak{c}}^p$ onto the $\text{ad}(\mathfrak{J})$ -eigenspace of maximum eigenvalue satisfies

$$\pi_{\mathfrak{M}^{p(10)}}(\widehat{\mathfrak{g}}^p \cap \mathfrak{u}^p) = \mathfrak{m}^{p(10)}$$

for all $p \geq 0$.

Two models \mathfrak{g} and \mathfrak{g}' of type respectively \mathfrak{m} and \mathfrak{m}' are called *isomorphic* if $\mathfrak{g}' = g \cdot \mathfrak{g}$ for some $g \in \text{Aut}(\mathfrak{c}, \mathfrak{J})$; if this is the case then $\mathfrak{m}' = g \cdot \mathfrak{m}$ too. A model not contained in any other model is called *maximal*.

We emphasize that we do not require $\mathfrak{J} \in \mathfrak{g}^0$ in general. We say that \mathfrak{g} has the *property* (\mathfrak{J}) when $\mathfrak{J} \in \mathfrak{g}^0$ and note that in this special case $\widehat{\mathfrak{g}}$ decomposes into $\text{ad}(\mathfrak{J})$ -eigenspaces, with the abstract core \mathfrak{m} actually contained in \mathfrak{g} . This property is a direct generalization of the “property (J)” introduced and studied in [19] for Levi-Tanaka algebras, see also [18]. We anticipate that we will encounter in Theorem 6.1 a model which does not satisfy property (\mathfrak{J}) : this fact says that the core \mathfrak{m} *cannot* always be realized simply as a subspace of \mathfrak{g} and it is also the main reason that motivated weaker property (iv) in Definition 4.1.

The following is the first main result on homogeneous CR manifolds. We remark that a CR subalgebra $(\mathfrak{g}, \mathfrak{q})$ of an holomorphically nondegenerate CR algebra $(\mathfrak{c}, \mathfrak{u})$ is not necessarily holomorphically nondegenerate (see [21]), in our setting this stems from (iii) and (iv), Definition 4.1.

Theorem 4.2. *Let $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ be a model of type \mathfrak{m} . Then:*

- (i) $\mathfrak{q} = \widehat{\mathfrak{g}} \cap \mathfrak{u}$ is a complex Lie subalgebra of $\widehat{\mathfrak{g}}$ with the compatible grading $\mathfrak{q} = \bigoplus_{p \geq -1} \mathfrak{q}^p$ with components $\mathfrak{q}^p = \widehat{\mathfrak{g}}^p \cap \mathfrak{u}^p$;
- (ii) the pair $(\mathfrak{g}, \mathfrak{q})$ is a CR algebra with real isotropy algebra $\mathfrak{g}_o = \mathfrak{g} \cap \mathfrak{u}$ which is \mathbb{Z} -graded in nonnegative degrees;
- (iii) \mathfrak{g} is finite dimensional and the associated (germ of) locally homogeneous CR manifold $M = G/G_o$ with algebra of infinitesimal CR automorphisms $(\mathfrak{g}, \mathfrak{q})$ is of hypersurface type and strongly regular of type \mathfrak{m} (in particular it is k -nondegenerate, $k = \text{ht}(\mathfrak{m}) + 2$).

Proof. Point (i) follows immediately as \mathfrak{q} is the intersection of two \mathbb{Z} -graded Lie subalgebras of $\widehat{\mathfrak{c}}$ and therefore a \mathbb{Z} -graded Lie subalgebra of $\widehat{\mathfrak{c}}$. The real isotropy algebra of the pair $(\mathfrak{g}, \mathfrak{q})$ is \mathbb{Z} -graded in nonnegative degrees,

$$\mathfrak{g}_o = \mathfrak{g} \cap \mathfrak{q} = \mathfrak{g} \cap \mathfrak{u} = \bigoplus_{p \geq 0} \mathfrak{g}^p \cap \mathfrak{u}^p,$$

and $\dim(\mathfrak{g}/\mathfrak{g}_o) < +\infty$ by (iii)-(iv) of Definition 4.1 and since \mathfrak{m} is finite dimensional by definition. This shows that $(\mathfrak{g}, \mathfrak{q})$ is a CR algebra.

To prove (iii) we first show that the q -th term, $q \geq 0$, of the Freeman sequence of $(\mathfrak{g}, \mathfrak{q})$ is given by

$$\mathfrak{q}_q = \bigoplus_{p \geq q} \mathfrak{q}^p \oplus \bigoplus_{0 \leq p \leq q-1} \mathfrak{q}^p \cap \bar{\mathfrak{q}}^p. \quad (4.1)$$

First note that each term \mathfrak{q}_q is \mathbb{Z} -graded, by its very definition (2.2) and the fact that $\bar{\mathfrak{q}}$ is \mathbb{Z} -graded. Conditions (i) and (iii) of Definition 4.1 imply then

$$\begin{aligned} \mathfrak{q}_0 &= \left\{ X \in \mathfrak{q} \mid [X, \bar{\mathfrak{q}}] \subset \mathfrak{q} + \bar{\mathfrak{q}} \right\} \\ &= \left\{ X \in \mathfrak{q} \mid [X, \bar{\mathfrak{q}}] \subset \bigoplus_{p \geq -1} \widehat{\mathfrak{g}}^p \right\} \\ &\subset \left\{ X \in \mathfrak{q} \mid [X, \mathfrak{c}^{-1(01)}] \subset \bigoplus_{p \geq -1} \widehat{\mathfrak{g}}^p \right\}, \end{aligned}$$

and hence $\mathfrak{q}_0 \subset \mathfrak{q} \cap \bigoplus_{p \geq 0} \widehat{\mathfrak{g}}^p = \bigoplus_{p \geq 0} \mathfrak{q}^p$ by the nondegeneracy of $\mathfrak{c}_- = \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1}$; the opposite

inclusion is immediate, hence $\mathfrak{q}_0 = \bigoplus_{p \geq 0} \mathfrak{q}^p$.

Assume now that (4.1) holds for any q strictly smaller than a positive integer $r + 1$; we want to show (4.1) for $q = r + 1$ too. The induction hypothesis and condition (iii) imply

$$\begin{aligned} \mathfrak{q}_{r+1} &= \left\{ X \in \mathfrak{q}_r \mid [X, \bar{\mathfrak{q}}] \subset \mathfrak{q}_r + \bar{\mathfrak{q}} \right\} \\ &= \left\{ X \in \bigoplus_{p \geq r} \mathfrak{q}^p \oplus \bigoplus_{0 \leq p \leq r-1} \mathfrak{q}^p \cap \bar{\mathfrak{q}}^p \mid [X, \bar{\mathfrak{q}}] \subset \bigoplus_{p \geq r} \mathfrak{q}^p + \bar{\mathfrak{q}} \right\} \\ &= \left\{ X \in \bigoplus_{p \geq r} \mathfrak{q}^p \oplus \bigoplus_{0 \leq p \leq r-1} \mathfrak{q}^p \cap \bar{\mathfrak{q}}^p \mid [X, \bar{\mathfrak{q}}] \subset \bigoplus_{p \geq r} \widehat{\mathfrak{g}}^p + \bar{\mathfrak{q}} \right\} \end{aligned}$$

and $\bigoplus_{p \geq r+1} \mathfrak{q}^p \oplus \bigoplus_{0 \leq p \leq r} \mathfrak{q}^p \cap \bar{\mathfrak{q}}^p \subset \mathfrak{q}_{r+1}$. We now prove the opposite inclusion.

By condition (iv) the natural projection $\pi_{\mathfrak{M}^{r(10)}} : \widehat{\mathfrak{c}}^r \longrightarrow \mathfrak{M}^{r(10)}$ yields an identification $\mathfrak{q}^r / \mathfrak{q}^r \cap \bar{\mathfrak{q}}^r \simeq \mathfrak{m}^{r(10)}$ and any $X \in \mathfrak{q}^r$ has a unique decomposition $X = X^{10} + X_o$ where $X^{10} = \pi_{\mathfrak{M}^{r(10)}}(X) \in \mathfrak{m}^{r(10)}$ and $X_o \in \mathfrak{u}^r \cap \bar{\mathfrak{u}}^r$. By Proposition 3.4 and the definition of core we have for all $v \in \mathfrak{c}^{-1(01)}$

$$\begin{aligned} [X, v] &= [X^{10}, v] + [X_o, v] \quad \text{where} \\ [X^{10}, v] &\in \mathfrak{m}^{r-1(10)} \quad \text{and} \quad [X_o, v] \in \bar{\mathfrak{u}}^{r-1} \end{aligned}$$

and $X^{10} = 0$ if and only if $[X^{10}, \mathfrak{c}^{-1(01)}] = 0$. This yields

$$\mathfrak{q}_{r+1} \subset \bigoplus_{p \geq r+1} \mathfrak{q}^p \oplus \bigoplus_{0 \leq p \leq r} \mathfrak{q}^p \cap \bar{\mathfrak{q}}^p$$

and hence (4.1) for all $q \geq 0$. The claims on the core and the k -nondegeneracy follow then from (3.10) and the identifications

$$\mathfrak{m}_x^{q(10)} \simeq \frac{\mathcal{F}_q^{10}|_x}{\mathcal{F}_{q+1}^{10}|_x} \simeq \mathfrak{q}_q / \mathfrak{q}_{q+1} \simeq \mathfrak{q}^q / \mathfrak{q}^q \cap \bar{\mathfrak{q}}^q \simeq \mathfrak{m}^{q(10)},$$

for all $q \geq 0$. In view of k -nondegeneracy and [3, Prop. 3.1] (see also [11, Remark, pag. 904] for more details), it follows also that \mathfrak{g} is finite-dimensional.

Finally, the (germ of) locally homogeneous CR manifold $M = G/G_o$ associated with the CR algebra $(\mathfrak{g}, \mathfrak{q})$ is of hypersurface type since

$$\begin{aligned} T_x^{\mathbb{C}} M &\simeq \widehat{\mathfrak{g}} / \mathfrak{q} \cap \bar{\mathfrak{q}}, \\ \mathcal{D}^{\mathbb{C}}|_x &\simeq \mathfrak{q} + \bar{\mathfrak{q}} / \mathfrak{q} \cap \bar{\mathfrak{q}}, \end{aligned}$$

by (2.1) and $\widehat{\mathfrak{g}}^p = \mathfrak{q}^p + \bar{\mathfrak{q}}^p$ for all $p \geq -1$; it is clearly strongly regular, by transitivity of the action of the Lie algebra \mathfrak{g} of infinitesimal CR automorphisms. This proves (iii). \square

Remark 4.3. It might be interesting to look for a purely algebraic proof of the fact that any model is finite-dimensional. In this paper we show the somehow weaker fact that most of the finite dimensional models \mathfrak{g} that we determine are maximal (see Theorem 5.3, Theorem 6.1).

Example 4.4 (*Levi-nondegenerate CR manifolds*).

A connected CR manifold $(M, \mathcal{D}, \mathcal{J})$ of dimension $2n + 1$ with first Levi form $\mathcal{L}^1 : \mathcal{D}^{10} \times \mathcal{D}^{01} \longrightarrow T^{\mathbb{C}} M / \mathcal{D}^{\mathbb{C}}$ nondegenerate at all points is strongly regular of type \mathfrak{m} with $\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1}$ given by the Heisenberg algebra of some signature $\text{sgn}(\mathfrak{J}) = (r, s)$, $n = r + s$. By the results of [33, 34, 19] (see also [35, §3]), for any signature there exists a unique maximal model, it is a grading $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{su}(r + 1, s + 1)$ with

$$\mathfrak{g}^p = \begin{cases} 0 & \text{for all } |p| > 2, \\ \mathbb{R} & \text{for } p = -2, \\ \mathbb{C}^n & \text{for } p = -1, \\ \mathfrak{u}(r, s) \oplus \mathbb{R}E & \text{for } p = 0, \\ (\mathbb{C}^n)^* & \text{for } p = 1, \\ \mathbb{R}^* & \text{for } p = 2. \end{cases}$$

In other words $\widehat{\mathfrak{g}}^0 = S^{1,1} \oplus \mathbb{C}E$, $\widehat{\mathfrak{g}}^1 = S^{1,0} \oplus S^{0,1}$ and $\widehat{\mathfrak{g}}^2 = S^0$ (this can also be checked directly using Proposition 3.2).

Example 4.5 (*CR manifolds of dimension 5*).

A 5-dimensional connected CR manifold $(M, \mathcal{D}, \mathcal{J})$ with k -nondegenerate Levi form, $k \geq 2$, is actually 2-nondegenerate at all points and strongly regular of type \mathfrak{m} where $\text{sgn}(\mathfrak{J}) = (1, 0)$ and

$$\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 \quad \text{with}$$

$$\mathfrak{m}^{-2} = \mathbb{R}, \quad \mathfrak{m}^{-1} = \mathbb{C}, \quad \mathfrak{m}^0 = \mathfrak{M}^0 = \Re(S^{2,0} \oplus S^{0,2}),$$

by a simple dimension argument applied to (3.11) with $n = 1$. By the results of [9, 10] there is a unique maximal model of type \mathfrak{m} , a grading $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of the simple Lie algebra $\mathfrak{g} = \mathfrak{so}(3, 2)$ with

$$\mathfrak{g}^p = \begin{cases} 0 & \text{for all } |p| > 2, \\ \mathbb{R} & \text{for } p = -2, \\ \mathbb{C} & \text{for } p = -1, \\ \mathfrak{sp}_2(\mathbb{R}) \oplus \mathbb{R}E & \text{for } p = 0, \\ \mathbb{C}^* & \text{for } p = 1, \\ \mathbb{R}^* & \text{for } p = 2, \end{cases}$$

(see also [26] and the description in [22, §3.2]). In other words

$$\widehat{\mathfrak{g}}^0 = \widehat{\mathfrak{c}}^0 = \widehat{\mathfrak{M}}^0 \oplus S^{1,1} \oplus \mathbb{C}E \quad \text{and}$$

$$\widehat{\mathfrak{g}}^1 = S^{1,0} \oplus S^{0,1}, \quad \widehat{\mathfrak{g}}^2 = S^0.$$

We remark that \mathfrak{g}^0 not only contains the 0-degree part $\mathfrak{u}(1) \oplus \mathbb{R}E$ of the real isotropy algebra but also the real part of the direct sum $\widehat{\mathfrak{M}}^0 = S^{2,0} \oplus S^{0,2}$ of the two $\text{ad}(\mathfrak{J})$ -eigenspaces in $\widehat{\mathfrak{c}}^0$ of extremal eigenvalues $\pm 2i$.

The main aim of §5 and §6 is to study 7-dimensional models and their associated homogeneous CR manifolds. In view of Definition 4.1 and Theorem 4.2 it is important to first classify the 7-dimensional abstract cores up to isomorphisms. This is the content of §4.2.

4.2. Classification of abstract cores in dimension 7. A simple argument yields three main classes of abstract cores associated with strongly regular 7-dimensional CR manifolds.

Class	\mathfrak{m}^{-2}	\mathfrak{m}^{-1}	\mathfrak{m}^0	\mathfrak{m}^1	\mathfrak{m}^p ($p > 1$)
(A)	\mathbb{R}	\mathbb{C}^3	0	0	0
(B)	\mathbb{R}	\mathbb{C}^2	\mathbb{C}	0	0
(C)	\mathbb{R}	\mathbb{C}	\mathbb{C}	\mathbb{C}	0

TABLE 1. The abstract cores $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ with $\dim(\mathfrak{m}) = 7$.

Note that the case $\mathfrak{m}^{-2} = \mathbb{R}$, $\mathfrak{m}^{-1} = \mathbb{C}$, $\mathfrak{m}^0 = \mathbb{C}^2$ and $\mathfrak{m}^p = 0$ for all $p > 0$ is not permissible by (3.11). Class (A) correspond to the Levi-nondegenerate case described in Example 4.4 and it is easy to see that there exists just one core in class (C), i.e. the “3-nondegenerate” core

$$\mathfrak{m}^{-2} = \mathfrak{c}^{-2} = \langle e^{-2} \rangle, \quad \mathfrak{m}^{-1} = \mathfrak{c}^{-1} = \langle e_1, e_2 \rangle \quad \text{and} \\ \mathfrak{m}^{0(10)} = \mathfrak{M}^{0(10)}, \quad \mathfrak{m}^{1(10)} = \mathfrak{M}^{1(10)}. \quad (4.2)$$

The description of the equivalence classes of cores in (B) is more involved and splits into $\text{sgn}(\mathfrak{J}) = (2, 0)$ and $\text{sgn}(\mathfrak{J}) = (1, 1)$. For our purposes it is convenient to consider a *complex-symplectic basis* $\{e_1, \dots, e_4\}$ of $\mathfrak{c}^{-1} \simeq \mathbb{R}^4$, that is a real basis satisfying

$$\begin{cases} B(e_i, e_{i+2}) = -B(e_{i+2}, e_i) = 1 & \text{for all } 1 \leq i \leq 2, \\ B(e_i, e_j) = 0 & \text{otherwise,} \\ \mathfrak{J}(e_i) = -e_{i+2}, \quad \mathfrak{J}(e_{i+2}) = e_i & \text{for all } 1 \leq i \leq r, \\ \mathfrak{J}(e_i) = e_{i+2}, \quad \mathfrak{J}(e_{i+2}) = -e_i & \text{for all } r+1 \leq i \leq 2. \end{cases} \quad (4.3)$$

We also record here, for later use in §5, that another kind of natural basis can be considered when $\text{sgn}(\mathfrak{J}) = (1, 1)$. In this case

$$\begin{cases} B(e_i, e_{i+2}) = -B(e_{i+2}, e_i) = 1 & \text{for all } 1 \leq i \leq 2, \\ B(e_i, e_j) = 0 & \text{otherwise,} \\ \mathfrak{J}(e_1) = -e_4, & \mathfrak{J}(e_4) = e_1, \\ \mathfrak{J}(e_2) = -e_3, & \mathfrak{J}(e_3) = e_2, \end{cases} \quad (4.4)$$

and we call such a basis a *complex-Witt basis* of \mathfrak{c}^{-1} . It is easy to see that the associated vectors

$$\begin{aligned} e'_1 &= \frac{e_1 + e_2}{\sqrt{2}}, & e'_2 &= \frac{e_1 - e_2}{\sqrt{2}}, \\ e'_3 &= \frac{e_3 + e_4}{\sqrt{2}}, & e'_4 &= \frac{e_3 - e_4}{\sqrt{2}}, \end{aligned} \quad (4.5)$$

constitute a complex symplectic basis $\{e'_1, \dots, e'_4\}$ of \mathfrak{c}^{-1} .

To classify the abstract core in class (B) of a fixed signature we first need some simple observations concerning the action of the Lie group $\text{Aut}(\mathfrak{c}, \mathfrak{J})$ (recall the discussion above Definition 3.5). First of all we note that any core $\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1} \oplus \mathfrak{m}^0 = \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1} \oplus \mathfrak{m}^0$ is fully determined by the complex line $\mathfrak{m}^{0(10)} \subset \mathfrak{M}^{0(10)} \simeq S^2\mathbb{C}^2$ given by the \mathfrak{J} -holomorphic part of $\widehat{\mathfrak{m}}^0 = \mathfrak{m}^{0(10)} \oplus \overline{\mathfrak{m}^{0(10)}}$. Furthermore the natural action of $\text{Aut}(\mathfrak{c}, \mathfrak{J})$ on $S^2\mathbb{C}^2$ factors through the quotient $K_{\sharp} = \text{Aut}(\mathfrak{c}, \mathfrak{J})/\mathbb{Z}_2$, namely to the Lie group $K_{\sharp} = \mathbb{C}^{\times} \cdot K$ associated with the connected and closed subgroup K of $\text{SL}_3(\mathbb{R})$ given by

$$\begin{aligned} K &= \text{SO}_3(\mathbb{R}) \simeq \text{SU}(2)/\mathbb{Z}_2 & \text{if } \text{sgn}(\mathfrak{J}) = (2, 0), \\ K &= \text{SO}^+(2, 1) \simeq \text{SU}(1, 1)/\mathbb{Z}_2 & \text{if } \text{sgn}(\mathfrak{J}) = (1, 1). \end{aligned}$$

More explicitly we fix basis (ϵ_{α}) of $\mathfrak{sl}_2(\mathbb{C}) \simeq S^2\mathbb{C}^2$ given by

$$\begin{aligned} \epsilon_1 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \simeq -\frac{1}{2}(e_1^{10} \odot e_1^{10} + e_2^{10} \odot e_2^{10}), \\ \epsilon_2 &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \simeq \frac{i}{2}(e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10}), \\ \epsilon_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \simeq -i(e_1^{10} \odot e_2^{10}) \end{aligned} \quad (4.6)$$

if $\text{sgn}(\mathfrak{J}) = (2, 0)$ and

$$\begin{aligned}\epsilon_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}(e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10}), \\ \epsilon_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\frac{i}{2}(e_1^{10} \odot e_1^{10} + e_2^{10} \odot e_2^{10}), \\ \epsilon_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -i(e_1^{10} \odot e_2^{10})\end{aligned}\tag{4.7}$$

if $\text{sgn}(\mathfrak{J}) = (1, 1)$, so that $\mathbb{R}^3 = \text{span}_{\mathbb{R}} \{\epsilon_1, \epsilon_2, \epsilon_3\}$ equals $\mathfrak{su}(2)$ and $\mathfrak{su}(1, 1)$, respectively. Using this identification, the action of $\text{Aut}(\mathfrak{c}, \mathfrak{J})$ on $S^2\mathbb{C}^2$ factors to the representation of K_{\sharp} on $V = \mathbb{C}^3 = \text{span}_{\mathbb{C}} \{\epsilon_1, \epsilon_2, \epsilon_3\}$

$$\rho : K_{\sharp} \longrightarrow \text{GL}(V)\tag{4.8}$$

given by the multiplications by the non-zero complex scalars and the \mathbb{C} -linear extension to V of the natural action of K on \mathbb{R}^3 :

$$\begin{aligned}(c, A) \cdot z &= c \cdot (Ax + iAy) \\ &= e^{i\vartheta} \cdot (\rho Ax + i\rho Ay) \\ &= (\rho \cos \vartheta \cdot Ax - \rho \sin \vartheta \cdot Ay) + i(\rho \cos \vartheta \cdot Ay + \rho \sin \vartheta \cdot Ax)\end{aligned}\tag{4.9}$$

where $c = \rho e^{i\vartheta} \in \mathbb{C}^{\times}$, $A \in K$ and $z = x + iy \in V$. It follows that two cores \mathfrak{m} and \mathfrak{m}' of the same signature are isomorphic if and only if corresponding elements $z, z' \in V^{\times}$ (determined up to nonzero complex scalars) lie on the same K_{\sharp} -orbit. In other words we consider the projective plane $X = \mathbb{P}^2(\mathbb{C})$ with the natural projective representation of K_{\sharp} and identify the orbit space

$$\begin{aligned}\mathbb{X} &= X/K_{\sharp} \\ &= \left\{ K_{\sharp} \cdot [z] \mid [z] \in X \right\}\end{aligned}$$

with V^{\times}/K_{\sharp} via the canonical equivariant projection from V^{\times} to X .

Let now $N_{\sharp} \subset K_{\sharp}$ be the stabilizer of $[z]$ and $\mathfrak{n}_{\sharp} = \text{Lie}(N_{\sharp})$ its associated Lie algebra. It is a subalgebra of $\text{Lie}(K_{\sharp}) = \mathbb{C} \oplus \mathfrak{su}(2)$ or $\mathbb{C} \oplus \mathfrak{su}(1, 1)$.

Definition 4.6. An orbit $K_{\sharp} \cdot [z]$ is called of *type* N_{\sharp} if $K_{\sharp} \cdot [z] \simeq K_{\sharp}/N_{\sharp}$.

To determine the K_{\sharp} -orbits on V it is also convenient, as an intermediate step in the proof of the following Proposition 4.8 and Proposition 4.10, to consider the subgroup $\tilde{K} = \mathbb{R}^{\times} \cdot K$ of K_{\sharp} and the restriction

$$\tilde{\rho} = \rho|_{\tilde{K}} : \tilde{K} \longrightarrow \text{GL}(V)\tag{4.10}$$

of the representation (4.8). Indeed $V \simeq \mathbb{R}^3 \oplus \mathbb{R}^3$ as a \tilde{K} -module, but not as a K_{\sharp} -module. Moreover $\mathbb{C}^{\times} \cdot \tilde{H} \subset N_{\sharp}$ where \tilde{H} is the stabilizer of z in \tilde{K} (we remark that this inclusion is in general proper, as there are elements $A \in K$ and $c \in \mathbb{C}^{\times}$ with the property that $A \cdot z = c \cdot z$).

Definition 4.7. An orbit $K_{\sharp} \cdot [z]$ and the associated equivalence class of cores is called *admissible* if $\mathfrak{n}_{\sharp} \neq \mathbb{C}$ (i.e., if the connected component of N_{\sharp} is not as smallest as possible).

We now deal with the two signatures separately.

4.2.1. *The case $\text{sgn}(\mathfrak{J}) = (2, 0)$.*

Let $\rho : K_{\sharp} = \mathbb{C}^{\times} \cdot K \longrightarrow \text{GL}(V)$ be given by the multiplications by \mathbb{C}^{\times} and the \mathbb{C} -linear extension to V of the natural action of $K = \text{SO}_3(\mathbb{R})$ on the Euclidean vector space $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$.

Proposition 4.8. *Every element $z \in V^{\times}$ is K_{\sharp} -related to one and only one of the canonical forms $\epsilon_1 + it\epsilon_2$ for some $t \in [0, 1]$. The corresponding K_{\sharp} -orbit is of type N_{\sharp} with $\mathfrak{n}_{\sharp} = \mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$ if $t = 0, 1$ and $\mathfrak{n}_{\sharp} = \mathbb{C}$ otherwise.*

Proof. Postponed to Appendix A. □

We can now directly apply this result to the classification of abstract cores.

Theorem 4.9. *Every 7-dimensional abstract core $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ of type (B) and $\text{sgn}(\mathfrak{J}) = (2, 0)$ is of the form*

$$\begin{aligned} \mathfrak{m}^p &= 0 \quad \text{for all } p \neq -2, -1, 0, \\ \mathfrak{m}^{-2} &= \mathfrak{c}^{-2} = \langle e^{-2} \rangle, \\ \mathfrak{m}^{-1} &= \mathfrak{c}^{-1} = \langle e_1, e_2, e_3, e_4 \rangle, \\ \mathfrak{m}^0 &= \Re(\mathfrak{m}^{0(10)} \oplus \overline{\mathfrak{m}^{0(10)}}), \end{aligned}$$

and isomorphic with one and only one of the canonical forms \mathfrak{m}_t in Table 2, $t \in [0, 1]$. Any 7-dimensional and strongly regular CR manifold $(M, \mathcal{D}, \mathcal{J})$ of type \mathfrak{m}_t is 2-nondegenerate and it is not (even locally) CR diffeomorphic to any other $(M', \mathcal{D}', \mathcal{J}')$ of type $\mathfrak{m}_{t'}$ if $t \neq t'$.

$\mathfrak{m}_t = \bigoplus \mathfrak{m}_t^p$ $t \in [0, 1]$	$\mathfrak{m}_t^{0(10)}$	\mathfrak{n}_{\sharp}
$t = 0, 1$	$(1+t)e_1^{10} \odot e_1^{10} + (1-t)e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$t \in (0, 1)$		\mathbb{C}

TABLE 2. The 7-dimensional cores of signature $(2, 0)$ up to isomorphism.

Proof. Two abstract cores \mathfrak{m} and \mathfrak{m}' are equivalent if and only if the corresponding elements $z, z' \in V^{\times}$ lie on the same K_{\sharp} -orbit. The first part of the theorem follows then from Proposition 4.8 and the second part is clear as $\text{ht}(\mathfrak{m}) = 0$ and the core is an invariant of a CR manifold. □

4.2.2. *The case $\text{sgn}(\mathfrak{J}) = (1, 1)$.*

In this case (4.8) is determined by the action of the connected Lorentz group $K = \text{SO}^+(2, 1)$ on the pseudo-Euclidean space $\mathbb{R}^{2,1}$ with the orthonormal basis (ϵ_{α}) such that $\langle \epsilon_1, \epsilon_1 \rangle =$

$\langle \epsilon_2, \epsilon_2 \rangle = -\langle \epsilon_3, \epsilon_3 \rangle = 1$. We recall that the orbits of K on $\mathbb{R}^{2,1} \setminus \{0\}$ split into five different types (see e.g. [13, 27]; the arguments are for $\mathbb{R}^{3,1}$ but extend easily to our case too):

$$\begin{aligned}
S_{r>0} &= \left\{ x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = r > 0 \right\} , \\
N^+ &= \left\{ x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = 0, x_3 > 0 \right\} , \\
N^- &= \left\{ x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = 0, x_3 < 0 \right\} , \\
S_{r<0}^+ &= \left\{ x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = r < 0, x_3 > 0 \right\} , \\
S_{r<0}^- &= \left\{ x \in \mathbb{R}^{2,1} \mid \langle x, x \rangle = r < 0, x_3 < 0 \right\} .
\end{aligned} \tag{4.11}$$

Any orbit of the first three types is diffeomorphic to $S^1 \times \mathbb{R}$ and of the form K/H where the stabilizer H is non-compact and conjugated in K to the subgroup

$$H \simeq \begin{cases} \text{the subgroup } \text{SO}^+(1,1) \text{ of boosts in the case } S_{r>0} , \\ \text{the subgroup } \mathbb{R} \text{ of null rotations in the case } N^\pm . \end{cases}$$

For example, the stabilizer of $\epsilon_1 + \epsilon_3$ is represented in the “mixed basis” $\{\epsilon_1 + \epsilon_3, \epsilon_2, \epsilon_1 - \epsilon_3\}$ by the one-parameter group of null rotations

$$\begin{pmatrix} 1 & -d/2 & -d^2/4 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} , \quad d \in \mathbb{R} .$$

Any orbit $S_{r<0}^\pm$ is of the form $K/H \simeq \mathbb{R}^2$ where $H \simeq \text{SO}_2(\mathbb{R})$ is compact subgroup of ordinary rotations. The orbits (4.11) are surfaces of transitivity for $\text{O}^+(2,1)$ too whereas those for the general Lorentz group $\text{O}(2,1)$ are of only three types, namely $S_{r>0}$, $N = N^+ \cup N^-$ and $S_{r<0} = S_{r<0}^+ \cup S_{r<0}^-$. These three surfaces are surfaces of transitivity for $\text{SO}(2,1)$ too.

It follows that the orbits of $\tilde{K} = \mathbb{R}^\times \cdot K$ on $\mathbb{R}^{2,1} \setminus \{0\}$ are exactly three:

$$S_{>0} = \bigcup S_{r>0} , \quad S_{<0} = \bigcup S_{r<0} \quad \text{and} \quad N ,$$

with only $S_{>0}$ connected. The following table gives representatives for these orbits together with the associated stabilizers $\tilde{H} \subset \tilde{K}$.

Orbit	Representative	\tilde{H}
$S_{>0}$	ϵ_1	$\text{SO}^+(1,1) \cup \text{SO}^+(1,1) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
N	$\epsilon_1 + \epsilon_3$	$\mathbb{R}^+ \ltimes \mathbb{R}$
$S_{<0}$	ϵ_3	$\text{SO}_2(\mathbb{R})$

TABLE 3. The orbits of $\tilde{K} = \mathbb{R}^\times \cdot \text{SO}^+(2,1)$ on $\mathbb{R}^{2,1} \setminus \{0\}$.

Here the subgroup $\mathbb{R}^+ \ltimes \mathbb{R}$ is the semidirect product of the subgroup \mathbb{R} of null rotations and the one parameter group of “dilations” given in the mixed basis by

$$\mathbb{R}^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix} \mid \rho > 0 \right\}.$$

A deeper analysis of the representation (4.8) gives the following.

Proposition 4.10. *Every element $z \in V^\times$ is K_\sharp -related to one and only one of the canonical forms:*

1. $\epsilon_1 + i\epsilon_2$ for $t \in [-1, 1]$;
2. $\epsilon_1 + \epsilon_3 + i(t(\epsilon_1 + \epsilon_3) + (\epsilon_1 - \epsilon_3))$ for $t \in \mathbb{R}$;
3. $\epsilon_1 + \epsilon_3 \pm i\epsilon_2$;
4. ϵ_3 ;
5. $\epsilon_1 + \epsilon_3$.

The corresponding K_\sharp -orbit is of type N_\sharp with

- $\mathfrak{n}_\sharp = \mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$ for $\epsilon_1 \pm i\epsilon_2$ and ϵ_3 ;
- $\mathfrak{n}_\sharp = \mathbb{C} \oplus \mathfrak{so}(1, 1)$ for ϵ_1 ;
- $\mathfrak{n}_\sharp = \mathbb{C} \oplus (\mathbb{R} \oplus \mathbb{R})$ for $\epsilon_1 + \epsilon_3$;
- $\mathfrak{n}_\sharp = \mathbb{C}$ otherwise.

Proof. Postponed to Appendix B. □

In the following $\{e_1, \dots, e_4\}$ is, as usual, a complex-symplectic basis of \mathfrak{c}^{-1} (and not a complex-Witt basis).

Theorem 4.11. *Every 7-dimensional abstract core $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ of type (B) and $\text{sgn}(\mathfrak{J}) = (1, 1)$ is of the form*

$$\begin{aligned} \mathfrak{m}^p &= 0 \text{ for all } p \neq -2, -1, 0, \\ \mathfrak{m}^{-2} &= \mathfrak{c}^{-2} = \langle e^{-2} \rangle, \\ \mathfrak{m}^{-1} &= \mathfrak{c}^{-1} = \langle e_1, e_2, e_3, e_4 \rangle, \\ \mathfrak{m}^0 &= \Re(\mathfrak{m}^{0(10)} \oplus \overline{\mathfrak{m}^{0(10)}}) \end{aligned}$$

and isomorphic with one and only one of the canonical forms \mathfrak{m}_t , $\tilde{\mathfrak{m}}_t$, \mathfrak{m}_\pm , $\mathfrak{m}_{<0}$ and $\mathfrak{m}_{\text{null}}$ in Table 4. Any 7-dimensional and strongly regular CR manifold $(M, \mathcal{D}, \mathcal{J})$ of type \mathfrak{m} where \mathfrak{m} is a core in Table 4 is 2-nondegenerate and it is not (even locally) CR diffeomorphic to any other $(M', \mathcal{D}', \mathcal{J}')$ whose core \mathfrak{m}' is also in Table 4 and different from \mathfrak{m} .

Proof. It is completely analogous to that of Theorem 4.9 and uses Proposition 4.10. □

Remark 4.12. We remark that Proposition 4.10 and Theorem 4.11 are not exhaustive of the topological structure of the moduli space \mathbb{X} of cores \mathfrak{m} of $\text{sgn}(\mathfrak{J}) = (1, 1)$ up to equivalences (this is due to the fact that $\text{SU}(1, 1)$ is not compact). One can show for instance that $\mathfrak{m}_{<0}$ is a point at infinity of the $\tilde{\mathfrak{m}}_t$ ’s whereas $\mathfrak{m}_{\text{null}}$ is arbitrarily close to $\tilde{\mathfrak{m}}_0$; we will not need this finer structure in this paper.

We see from Table 2 and Table 4 that there are precisely seven different classes of abstract cores $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ with $\dim(\mathfrak{m}) = 7$ and $\text{ht}(\mathfrak{m}) = 0$ which are admissible (that is \mathfrak{n}_\sharp properly contains \mathbb{C}). The next section deals with the corresponding models.

$\mathfrak{m}_t = \bigoplus \mathfrak{m}_t^p$ $t \in [-1, 1]$	$\mathfrak{m}_t^{0(10)}$	$\mathfrak{n}_\#$
$t = \pm 1$	$(1+t)e_1^{10} \odot e_1^{10} + (t-1)e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$t = 0$		$\mathbb{C} \oplus \mathfrak{so}(1, 1)$
$t \neq \pm 1, 0$		\mathbb{C}
$\tilde{\mathfrak{m}}_t = \bigoplus \tilde{\mathfrak{m}}_t^p$ $t \in \mathbb{R}$	$\tilde{\mathfrak{m}}_t^{0(10)}$	$\mathfrak{n}_\#$
$t \in \mathbb{R}$	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} + 2(t-1)e_1^{10} \odot e_2^{10}$ $+ i((1+t)e_1^{10} \odot e_1^{10} - (1+t)e_2^{10} \odot e_2^{10} - 2e_1^{10} \odot e_2^{10})$	\mathbb{C}
$\mathfrak{m}_\pm = \bigoplus \mathfrak{m}_\pm^p$	$\mathfrak{m}_\pm^{0(10)}$	$\mathfrak{n}_\#$
$t = \pm 1$	$(1+t)e_1^{10} \odot e_1^{10} + (-1+t)e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$	\mathbb{C}
$\mathfrak{m}_{<0} = \bigoplus \mathfrak{m}_{<0}^p$	$\mathfrak{m}_{<0}^{0(10)}$	$\mathfrak{n}_\#$
	$e_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$\mathfrak{m}_{\text{null}} = \bigoplus \mathfrak{m}_{\text{null}}^p$	$\mathfrak{m}_{\text{null}}^{0(10)}$	$\mathfrak{n}_\#$
	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus (\mathbb{R} \in \mathbb{R})$

TABLE 4. The 7-dimensional cores of signature $(1, 1)$ up to isomorphism.

5. MODELS FOR 7-DIMENSIONAL 2-NONDEGENERATE CR MANIFOLDS

The main aim of this section is to prove the following.

Theorem 5.1. *For each of the seven admissible cores $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ of $\dim(\mathfrak{m}) = 7$, $\text{ht}(\mathfrak{m}) = 0$ there exists an associated model $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of type \mathfrak{m} and a 7-dimensional 2-nondegenerate homogeneous CR manifold $M = G/G_o$ which is globally defined. Homogeneous CR manifolds associated with different \mathfrak{m} are not, even locally, CR diffeomorphic one to the other.*

To prove it, we first need to briefly recall the description of the \mathbb{Z} -gradings of the semisimple Lie algebras (see e.g. [14, 37, 8]).

Let \mathfrak{g} be a complex semisimple Lie algebra. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system and by

$$\mathfrak{g}^\alpha = \left\{ X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \right\}$$

the associated root space of $\alpha \in \Delta$. Let $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}$ be the real subspace where all the roots are real valued; any element $\lambda \in (\mathfrak{h}_{\mathbb{R}}^*)^* \simeq \mathfrak{h}_{\mathbb{R}}$ with $\lambda(\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$ defines a grading $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ on \mathfrak{g} by setting:

$$\begin{cases} \mathfrak{g}^0 = \mathfrak{h} \oplus \sum_{\substack{\alpha \in \Delta \\ \lambda(\alpha)=0}} \mathfrak{g}^\alpha, \\ \mathfrak{g}^p = \sum_{\substack{\alpha \in \Delta \\ \lambda(\alpha)=p}} \mathfrak{g}^\alpha, \end{cases} \quad \text{for all } p \in \mathbb{Z}^\times,$$

and all possible gradings of \mathfrak{g} are of this form, for some choice of \mathfrak{h} and λ . We will refer to $\lambda(\alpha)$ as the *degree* of the root α .

There exists a set of positive roots $\Delta^+ \subset \Delta$ such that λ is dominant, i.e. $\lambda(\alpha) \geq 0$ for all $\alpha \in \Delta^+$. The depth of \mathfrak{g} is the degree of the maximal root and is also equal to the height of \mathfrak{g} . Let Π be the set of positive simple roots, which we identify with the nodes of the Dynkin diagram. A grading is fundamental if and only if $\lambda(\alpha) \in \{0, 1\}$ for all $\alpha \in \Pi$. We denote a fundamental grading of a Lie algebra \mathfrak{g} by marking with a cross the nodes of the Dynkin diagram of \mathfrak{g} corresponding to simple roots α with $\lambda(\alpha) = 1$.

The Lie subalgebra \mathfrak{g}^0 is reductive; the Dynkin diagram of its semisimple ideal is obtained from the Dynkin diagram of \mathfrak{g} by removing all crossed nodes, and any line issuing from them.

A routine examination of the Dynkin diagrams of complex Lie algebras together with e.g. [5, Prop. 3.2.4] implies that Table 5 below lists all the complex graded semisimple Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ which satisfy:

- (i) the depth $d(\mathfrak{g}) = 2$,
- (ii) $\dim \mathfrak{g}^{-2} = 1$,
- (iii) $\dim \mathfrak{g}^{-1} = 4$;
- (iv) \mathfrak{g}_- is nondegenerate.

Any such \mathfrak{g} is a simple graded subalgebra of the complex contact algebra $\widehat{\mathfrak{c}}$ of degree $n = 2$.

\mathfrak{g}	Grading	\mathfrak{g}^{-2}	\mathfrak{g}^{-1}	\mathfrak{g}^0
A_3		\mathbb{C}	$\mathbb{C}^2 \oplus (\mathbb{C}^2)^*$	$\mathfrak{gl}_2(\mathbb{C}) \oplus \mathbb{C}$
C_3		\mathbb{C}	\mathbb{C}^4	$\mathfrak{sp}_4(\mathbb{C}) \oplus \mathbb{C}$
G_2		\mathbb{C}	$S^3\mathbb{C}^2$	$\mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}$

TABLE 5. Some \mathbb{Z} -gradings $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of complex simple Lie algebras with the associated representation of \mathfrak{g}^0 on \mathfrak{g}_- .

We now recall the description of simple real graded Lie algebras. Let \mathfrak{g} be a real simple Lie algebra. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a maximal abelian subspace $\mathfrak{h}_o \subset \mathfrak{p}$ and a maximal torus \mathfrak{h}_\bullet in the centralizer of \mathfrak{h}_o in \mathfrak{k} . Then $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_o$ is a maximally noncompact Cartan subalgebra of \mathfrak{g} .

Denote by $\Delta = \Delta(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$ the root system of $\widehat{\mathfrak{g}}$ and by $\mathfrak{h}_\mathbb{R} = i\mathfrak{h}_\bullet \oplus \mathfrak{h}_o \subset \widehat{\mathfrak{h}}$ the real subspace where all the roots have real values. Conjugation $\sigma : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$ of $\widehat{\mathfrak{g}}$ with respect to the real form \mathfrak{g} leaves $\widehat{\mathfrak{h}}$ invariant and induces an involution $\alpha \mapsto \bar{\alpha}$ on $\mathfrak{h}_\mathbb{R}^*$, trasforming roots into roots. We say that a root α is compact if $\bar{\alpha} = -\alpha$ and denote by Δ_\bullet the set of compact roots. There exists a set of positive roots $\Delta^+ \subset \Delta$, with corresponding system of simple roots Π , and an involutive automorphism $\varepsilon : \Pi \rightarrow \Pi$ of the Dynkin diagram of $\widehat{\mathfrak{g}}$ such that $\varepsilon(\Pi \setminus \Delta_\bullet) \subseteq \Pi \setminus \Delta_\bullet$ and

$$\begin{aligned} \bar{\alpha} &= -\alpha \quad \text{for all } \alpha \in \Pi \cap \Delta_\bullet, \\ \bar{\alpha} &= \varepsilon(\alpha) + \sum_{\beta \in \Pi \cap \Delta_\bullet} b_{\alpha, \beta} \beta \quad \text{for all } \alpha \in \Pi \setminus \Delta_\bullet. \end{aligned}$$

The Satake diagram of \mathfrak{g} is the Dynkin diagram of $\widehat{\mathfrak{g}}$ with the following additional information:

- (1) nodes in $\Pi \cap \Delta_\bullet$ are painted black;
- (2) if $\alpha \in \Pi \setminus \Delta_\bullet$ and $\varepsilon(\alpha) \neq \alpha$ then α and $\varepsilon(\alpha)$ are joined by a curved arrow.

A list of Satake diagrams can be found in e.g. [5, 14, 24].

Let $\lambda \in (\mathfrak{h}_\mathbb{R}^*)^* \simeq \mathfrak{h}_\mathbb{R}$ be an element such that the induced grading on $\widehat{\mathfrak{g}}$ is fundamental. Then the grading on $\widehat{\mathfrak{g}}$ induces a grading on \mathfrak{g} if and only if $\bar{\lambda} = \lambda$, or equivalently the following two conditions on the set

$$\Phi = \left\{ \alpha \in \Pi \mid \lambda(\alpha) = 1 \right\}$$

are satisfied:

- (1) $\Phi \cap \Delta_\bullet = \emptyset$;
- (2) if $\alpha \in \Phi$ then $\varepsilon(\alpha) \in \Phi$.

We indicate the grading on a real Lie algebra by crossing all nodes in Φ . In the real case too the Lie subalgebra \mathfrak{g}^0 is reductive and the Satake diagram of its semisimple ideal is obtained from the Satake diagram of \mathfrak{g} .

Table 6 lists all the real graded simple Lie algebras $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ such that the induced grading on $\widehat{\mathfrak{g}}$ is as in Table 5.

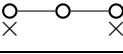
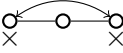

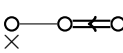

\mathfrak{g}	Grading	\mathfrak{g}^{-2}	\mathfrak{g}^{-1}	\mathfrak{g}^0
$\mathfrak{sl}_4(\mathbb{R})$		\mathbb{R}	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{gl}_2(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{su}(2, 2)$		\mathbb{R}	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{gl}_2(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{su}(1, 3)$		\mathbb{R}	$\mathbb{R}^2 \oplus (\mathbb{R}^2)^*$	$\mathfrak{u}(2) \oplus \mathbb{R}$
$\mathfrak{sp}_6(\mathbb{R})$		\mathbb{R}	\mathbb{R}^4	$\mathfrak{sp}_4(\mathbb{R}) \oplus \mathbb{R}$
G_2 split		\mathbb{R}	$S^3\mathbb{R}^2$	$\mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}$

TABLE 6. Some \mathbb{Z} -gradings $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of real simple Lie algebras with the associated representation of \mathfrak{g}^0 on \mathfrak{g}_- .

Let now $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ be a model of type \mathfrak{m} with $\dim(\mathfrak{m}) = 7$ and $\text{ht}(\mathfrak{m}) = 0$ that in addition is a simple Lie algebra satisfying property (\mathfrak{J}) (i.e. $\mathfrak{J} \in \mathfrak{g}^0$). By the discussion above the grading $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ is necessarily as in Table 6. It is convenient to introduce a Cartan subalgebra which, in general, is not maximally noncompact.

Definition 5.2. A Cartan subalgebra \mathfrak{h} of \mathfrak{g} is called *adapted* if it contains the grading element E of $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ and \mathfrak{J} .

Adapted Cartan subalgebras always exist since E and \mathfrak{J} are semisimple elements and $[E, \mathfrak{J}] = 0$. Moreover any adapted Cartan subalgebra \mathfrak{h} satisfies $\mathfrak{h} \subset \mathfrak{g}^0$ and decomposes into $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ where

$$\begin{aligned} \mathfrak{h}_\bullet &= \left\{ H \in \mathfrak{h} \mid \text{all eigenvalues of } \text{ad } H \text{ are purely imaginary} \right\}, \\ \mathfrak{h}_\circ &= \left\{ H \in \mathfrak{h} \mid \text{all eigenvalues of } \text{ad } H \text{ are real} \right\}. \end{aligned}$$

Clearly $\mathfrak{J} \in \mathfrak{h}_\bullet$ and $E \in \mathfrak{h}_\circ$. From now on \mathfrak{h} is an adapted Cartan subalgebra.

We fix an Hermitian form $h(t, s) = \bar{t}^t \mathcal{I} s$ on \mathbb{C}^4 and identify the associated special unitary Lie algebra \mathfrak{g} with the Lie algebra of trace-free complex matrices satisfying $\overline{A}^t \mathcal{I} + \mathcal{I} A = 0$; we follow the conventions of [30]:

$$(i) \quad \mathcal{I} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) \quad \text{and}$$

$$\mathfrak{g} = \mathfrak{su}(1, 3) = \left\{ \left(\begin{array}{c|c} -\text{tr } D & B \\ \hline \overline{B}^t & D \end{array} \right) \mid B \in \text{Mat}(1, 3; \mathbb{C}) \text{ and } D \in \mathfrak{u}(3) \right\} ;$$

$$(ii) \quad \mathcal{I} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right) \quad \text{and}$$

$$\mathfrak{g} = \mathfrak{su}(2, 2) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline \overline{B}^t & D \end{array} \right) \mid B \in \text{Mat}(2, 2; \mathbb{C}), A = \begin{pmatrix} u_1 & a \\ -\overline{a} & u_2 \end{pmatrix}, \right. \\ \left. D = \begin{pmatrix} u_3 & d \\ -\overline{d} & u_4 \end{pmatrix} \text{ where } u_i \in i\mathbb{R}, \sum u_i = 0 \right\} .$$

Theorem 5.3. *Let \mathfrak{g} be a model of type \mathfrak{m} , $\dim(\mathfrak{m}) = 7$ and $\text{ht}(\mathfrak{m}) = 0$, which in addition is a semisimple real Lie algebra and satisfies property (\mathfrak{J}) . Then:*

- (i) \mathfrak{g} is simple and isomorphic to $\mathfrak{sl}_4(\mathbb{R})$, $\mathfrak{su}(1, 3)$ or $\mathfrak{su}(2, 2)$ with the \mathbb{Z} -grading $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ given in Table 6;
- (ii) the complex structure \mathfrak{J} is described in Table 8 below;
- (iii) there always exists an associated 7-dimensional and 2-nondegenerate homogeneous CR manifold $M = G/G_o$ which is globally defined (i.e. G_o is closed in G). It is of type \mathfrak{m} where the signature $\text{sgn}(\mathfrak{J})$ of \mathfrak{J} and the equivalence class $[\mathfrak{m}^{0(10)}]$ of $\mathfrak{m}^{0(10)}$ are (recall Tables 2 and 4):

\mathfrak{g}	$\text{sgn}(\mathfrak{J})$	$[\mathfrak{m}^{0(10)}]$	\mathfrak{n}_{\sharp}
$\mathfrak{sl}_4(\mathbb{R})$	(1, 1)	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}(1, 1)$
$\mathfrak{su}(1, 3)$	(1, 1)	$e_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
$\mathfrak{su}(2, 2)$	(2, 0)	$e_1^{10} \odot e_1^{10} + e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$

TABLE 7.

- (iv) the adapted Cartan subalgebra \mathfrak{h} of \mathfrak{g} equals \mathfrak{n}_{\sharp} in all cases (recall Definition 4.6);
- (v) the model \mathfrak{g} is maximal;
- (vi) for completeness we also give the other terms (2.2) of the Freeman sequence of $(\mathfrak{g}, \mathfrak{q})$:

$$\mathfrak{q}_0 = \widehat{\mathfrak{g}}^2 \oplus \widehat{\mathfrak{g}}^1 \oplus \widehat{\mathfrak{h}} \oplus \mathfrak{g}^{-\alpha_2}, \quad \mathfrak{q}_1 = \widehat{\mathfrak{g}}^2 \oplus \widehat{\mathfrak{g}}^1 \oplus \widehat{\mathfrak{h}} = \mathfrak{q} \cap \overline{\mathfrak{q}},$$

with α_2 the simple root associated to the middle node in the Dynkin diagram of $\mathfrak{sl}_4(\mathbb{C})$.

\mathfrak{g}	$\text{sgn}(\mathfrak{J})$	E^2	E_1^{10}	E_2^{10}	E	\mathfrak{J}	H	$e^{0(10)}$	e_1^{10}	e_2^{10}	e^{-2}
			E_1^{01}	E_2^{01}				$e^{0(01)}$	e_1^{01}	e_2^{01}	
$\mathfrak{sl}_4(\mathbb{R})$	(1, 1)	$4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & -i & 0 & 0 \\ -i & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \end{pmatrix}$	$4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$
$\mathfrak{su}(1, 3)$	(1, 1)	$2 \begin{pmatrix} i & -i & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$2 \begin{pmatrix} i & i & 0 & 0 \\ -i & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$
$\mathfrak{su}(2, 2)$	(2, 0)	$2 \begin{pmatrix} i & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$2 \begin{pmatrix} i & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

TABLE 8. The explicit decomposition of \mathfrak{g} in terms of an adapted Cartan subalgebra $\mathfrak{h} = \{E, \mathfrak{J}, H\}$, a basis $\{e_1, \dots, e_4\}$ of \mathfrak{g}^{-1} which is complex-symplectic for $\mathfrak{g} = \mathfrak{su}(1, 3)$, $\mathfrak{su}(2, 2)$ and complex-Witt for $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$, and, finally, a generator $e^{0(10)}$ of $\mathfrak{m}^{0(10)}$ and a basis of elements “ E ’s” of the positive part $\mathfrak{g}^2 \oplus \mathfrak{g}^1$ of \mathfrak{g} .

Proof. We first show by contradiction that \mathfrak{g} is not $\mathfrak{sp}_6(\mathbb{R})$ or G_2 split.

If $\mathfrak{g} = \mathfrak{sp}_6(\mathbb{R})$ then $\mathfrak{g}^0 = \mathfrak{c}^0 \supset \Re(S^{2,0} \oplus S^{0,2})$ and $\dim(\mathfrak{m}) \geq 11$, a contradiction. Let \mathfrak{g} be G_2 split and $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ an adapted Cartan subalgebra of \mathfrak{g} with associated root system $\Delta = \Delta(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$; each root space $\mathfrak{g}^\alpha \subset \widehat{\mathfrak{g}}$ is in particular \mathfrak{J} -stable. Let also $\Delta^+ \subset \Delta$ be a set of positive roots such that λ is dominant and $\Pi = \{\alpha_1, \alpha_2\}$ the associated system of simple roots satisfying $\lambda(\alpha_1) = 0$ and $\lambda(\alpha_2) = 1$ (recall Table 5).

The roots $\alpha \in \Delta$ with $\lambda(\alpha) = -1$ are $-\alpha_2, -\alpha_1 - \alpha_2, -3\alpha_1 - \alpha_2, -2\alpha_1 - \alpha_2$ and the unique root with $\lambda(\alpha) = -2$ is $\alpha = -3\alpha_1 - 2\alpha_2$. By nondegeneracy of $\widehat{\mathfrak{g}}_-$, two possibilities may occur in $\widehat{\mathfrak{g}}^{-1} = \mathfrak{g}^{-1(10)} \oplus \mathfrak{g}^{-1(01)}$:

$\mathfrak{g}^{-1(10)}$	$\mathfrak{g}^{-1(01)}$	$\widehat{\mathfrak{g}}^0$
$\mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{-(\alpha_1 + \alpha_2)}$	$\mathfrak{g}^{-(3\alpha_1 + \alpha_2)} \oplus \mathfrak{g}^{-(2\alpha_1 + \alpha_2)}$	$\widehat{\mathfrak{h}} \oplus \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{-\alpha_1}$
$\mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{-(2\alpha_1 + \alpha_2)}$	$\mathfrak{g}^{-(3\alpha_1 + \alpha_2)} \oplus \mathfrak{g}^{-(\alpha_1 + \alpha_2)}$	$\widehat{\mathfrak{h}} \oplus \mathfrak{g}^{\alpha_1} \oplus \mathfrak{g}^{-\alpha_1}$

TABLE 9.

A direct inspection of the adjoint action of \mathfrak{g}^{α_1} on $\mathfrak{g}^{-1(10)}$ and $\mathfrak{g}^{-1(01)}$ yields that \mathfrak{g}^{α_1} is not \mathfrak{J} -stable in both cases, a contradiction.

Let now \mathfrak{g} be $\mathfrak{sl}_4(\mathbb{R})$, $\mathfrak{su}(1, 3)$ or $\mathfrak{su}(2, 2)$, $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ an adapted Cartan subalgebra with root system Δ and associated system $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ of simple roots satisfying $\lambda(\alpha_1) = \lambda(\alpha_3) = 1$ and $\lambda(\alpha_2) = 0$.

Since $\lambda(\alpha_1 + \alpha_2 + \alpha_3) = 2$, two possibilities occur in $\widehat{\mathfrak{g}}^{-1} = \mathfrak{g}^{-1(10)} \oplus \mathfrak{g}^{-1(01)}$:

$\mathfrak{g}^{-1(10)}$	$\mathfrak{g}^{-1(01)}$	$\widehat{\mathfrak{g}}^0$
$\mathfrak{g}^{-(\alpha_1 + \alpha_2)} \oplus \mathfrak{g}^{-(\alpha_2 + \alpha_3)}$	$\mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{-\alpha_3}$	$\widehat{\mathfrak{h}} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_2}$
$\mathfrak{g}^{-\alpha_3} \oplus \mathfrak{g}^{-(\alpha_2 + \alpha_3)}$	$\mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{-(\alpha_1 + \alpha_2)}$	$\widehat{\mathfrak{h}} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_2}$

TABLE 10.

In the second case $\widehat{\mathfrak{g}}^0 = S^{1,1} \oplus \mathbb{C}E$ and hence $\dim(\mathfrak{m}) = 5$, a contradiction.

In the first case $\mathfrak{g}^{-\alpha_2} \subset \mathfrak{M}^{0(10)}$ and $\mathfrak{g}^{\alpha_2} \subset \mathfrak{M}^{0(01)}$ and there is not a contradiction. To check that this case does actually occur we need to identify explicitly the adapted Cartan subalgebras of $\mathfrak{su}(1, 3)$, $\mathfrak{su}(2, 2)$ and $\mathfrak{sl}_4(\mathbb{R})$; we rely upon the classification, up to conjugation, of the Cartan subalgebras of the simple real Lie algebras given in [30].

If $\mathfrak{g} = \mathfrak{su}(1, 3)$ there are two Cartan subalgebras, the compact one (clearly not adapted) and the subalgebra $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ with $\dim \mathfrak{h}_\bullet = 2$, $\dim \mathfrak{h}_\circ = 1$:

$$\mathfrak{h}_\bullet = \left\{ H = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & u_4 \end{pmatrix} \mid u_i \in i\mathbb{R} \text{ and } \operatorname{tr} H = 0 \right\},$$

$$\mathfrak{h}_\circ = \left\{ H = \begin{pmatrix} 0 & h_1 & 0 & 0 \\ h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid h_1 \in \mathbb{R} \right\}.$$

Let $\{E, \mathfrak{J}, H\}$ be the basis of \mathfrak{h} given in Table 8; $E \in \mathfrak{h}_\circ$ and coincides with the grading element of the grading of $\mathfrak{su}(1, 3)$ in Table 6, the set $\{\mathfrak{J}, H\}$ is a basis of \mathfrak{h}_\bullet . Moreover $\operatorname{ad}(\mathfrak{J})|_{\mathfrak{g}^{-1}}$ and $\operatorname{ad}(H)|_{\mathfrak{g}^{-1}}$ are two linearly independent and commuting complex structures on \mathfrak{g}^{-1} but the eigenspace decomposition of $\operatorname{ad}(H)|_{\mathfrak{g}^{-1}}$ is as in the second case of Table 10 and hence not permissible.

It follows from this observation, Table 8 and the decomposition of $\mathfrak{su}(1, 3)$ under the adjoint action of \mathfrak{J} that

$$\begin{aligned} \mathfrak{g}^{-\alpha_1} &= \langle e_1^{01} \rangle, & \mathfrak{g}^{-\alpha_3} &= \langle e_2^{01} \rangle, \\ \mathfrak{g}^{-(\alpha_1+\alpha_2)} &= \langle e_2^{10} \rangle, & \mathfrak{g}^{-(\alpha_2+\alpha_3)} &= \langle e_1^{10} \rangle, \\ \mathfrak{g}^{-\alpha_2} &= \mathfrak{m}^{0(10)} = \langle e^{0(10)} \rangle, & \mathfrak{g}^{\alpha_2} &= \mathfrak{m}^{0(01)} = \langle e^{0(01)} \rangle, \end{aligned} \tag{5.1}$$

and $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ is a model of type \mathfrak{m} .

If $\mathfrak{g} = \mathfrak{su}(2, 2)$ there are three Cartan subalgebras, the compact one (not adapted), the one with $\dim \mathfrak{h}_\bullet = 1$ and $\dim \mathfrak{h}_\circ = 2$ and finally the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ with $\dim \mathfrak{h}_\bullet = 2$, $\dim \mathfrak{h}_\circ = 1$:

$$\mathfrak{h}_\bullet = \left\{ H = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & u_4 \end{pmatrix} \mid u_i \in i\mathbb{R} \text{ and } \operatorname{tr} H = 0 \right\},$$

$$\mathfrak{h}_\circ = \left\{ H = \begin{pmatrix} 0 & 0 & h_1 & 0 \\ 0 & 0 & 0 & 0 \\ h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid h_1 \in \mathbb{R} \right\}.$$
(5.2)

The Cartan subalgebra with $\dim \mathfrak{h}_\bullet = 1$ is not adapted, since in that case \mathfrak{h}_\bullet is generated by a semisimple element with eigenvalues $0, \pm i$. The subalgebra (5.2) is adapted and gives rise to a model, using (5.1), Table 8 and an argument analogous to the previous case.

If $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$ there are three Cartan subalgebras, the vectorial one (not adapted), the one with $\dim \mathfrak{h}_\bullet = 2$ and $\dim \mathfrak{h}_\circ = 1$ and finally the Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_\bullet \oplus \mathfrak{h}_\circ$ with $\dim \mathfrak{h}_\bullet = 1$

and $\dim \mathfrak{h}_\circ = 2$:

$$\begin{aligned} \mathfrak{h}_\bullet &= \left\{ H = \begin{pmatrix} 0 & -h_2 & 0 & 0 \\ h_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid h_2 \in \mathbb{R} \right\}, \\ \mathfrak{h}_\circ &= \left\{ H = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_1 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix} \mid h_i \in \mathbb{R} \text{ and } \operatorname{tr} H = 0 \right\}. \end{aligned} \quad (5.3)$$

The Cartan with $\dim \mathfrak{h}_\circ = 1$ is not adapted, as in that case \mathfrak{h}_\circ is generated by a semisimple element with eigenvalues $0, \pm 1$. The Cartan (5.3) is adapted. To see it note that the set $\{E, H\}$ as in Table 8 is a basis of \mathfrak{h}_\circ consisting of grading elements of two inequivalent gradings. The element corresponding to the grading of Table 6 is E . We can therefore argue as in previous cases once we note that the decomposition of $\mathfrak{sl}_4(\mathbb{R})$ under the adjoint action of the generator \mathfrak{J} of \mathfrak{h}_\bullet is in this case

$$\begin{aligned} \mathfrak{g}^{-\alpha_1} &= \langle e_1^{01} \rangle, & \mathfrak{g}^{-\alpha_3} &= \langle e_2^{01} \rangle, \\ \mathfrak{g}^{-(\alpha_1+\alpha_2)} &= \langle e_1^{10} \rangle, & \mathfrak{g}^{-(\alpha_2+\alpha_3)} &= \langle e_2^{10} \rangle, \\ \mathfrak{g}^{-\alpha_2} &= \mathfrak{m}^{0(10)} = \langle e^{0(10)} \rangle, & \mathfrak{g}^{\alpha_2} &= \mathfrak{m}^{0(01)} = \langle e^{0(01)} \rangle, \end{aligned} \quad (5.4)$$

where $\{e_1, \dots, e_4\}$ is a *complex-Witt basis*. This proves (i) and (ii).

The second part of (iii) follows from Theorem 4.2 and $\operatorname{sgn}(\mathfrak{J})$ from (3.2) and (4.3)-(4.4). A direct inspection of the adjoint action of $e^{0(10)}$ on $\mathfrak{g}^{-1(01)}$ and equation (3.3) give the component $\mathfrak{m}^{0(10)}$ of the core for $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(1, 3)$; in the first case we need to note

$$\mathfrak{m}^{0(10)} = \langle e_1^{10} \odot e_2^{10} \rangle \simeq \langle e_1^{10} \odot e_1^{10} + e_2^{10} \odot e_2^{10} \rangle.$$

If $\mathfrak{g} = \mathfrak{sl}_4(\mathbb{R})$ then $\mathfrak{m}^{0(10)} = \langle (e'_1)^{10} \odot (e'_1)^{10} - (e'_2)^{10} \odot (e'_2)^{10} \rangle$ w.r.t. the basis (4.5) associated to the complex-Witt basis of Table 8. The Lie algebra \mathfrak{n}_\sharp of the stabilizer of $\mathfrak{m}^{0(10)}$ is in Theorem 4.9 and Theorem 4.11.

The first part of (iii) follows from the following argument. Note that the complex subalgebra $\mathfrak{q} = \widehat{\mathfrak{g}} \cap \mathfrak{u}$ of the CR algebra $(\mathfrak{g}, \mathfrak{q})$ associated with the model \mathfrak{g} is

$$\mathfrak{q} = \widehat{\mathfrak{g}}^2 + \widehat{\mathfrak{g}}^1 + \widehat{\mathfrak{h}} + \mathfrak{g}^{-\alpha_2} + \mathfrak{g}^{-(\alpha_1+\alpha_2)} + \mathfrak{g}^{-(\alpha_2+\alpha_3)}.$$

It is the maximal 11-dimensional parabolic subalgebra of $\widehat{\mathfrak{g}}$ which corresponds, for an appropriate choice of system of simple roots, to the nonnegatively graded part of the \mathbb{Z} -grading of $\widehat{\mathfrak{g}}$ associated with the simply crossed Dynkin diagram $\circ - \times - \circ$ (a word of caution: the \mathbb{Z} -grading of Table 5 used in the construction of the models is different and associated with the 10-dimensional parabolic subalgebra $\widehat{\mathfrak{g}}_{\geq 0} = \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^1 \oplus \widehat{\mathfrak{g}}^2$). In particular $(\mathfrak{g}, \mathfrak{q})$ is a *parabolic CR algebra* in the sense of [21] and hence there always exists an associated globally defined homogeneous CR manifold $M = G/G_o$. This proves (iii).

Now $\mathfrak{h} = \mathfrak{n}_\sharp$ follows from $\dim \mathfrak{h} = \dim \mathfrak{n}_\sharp = 3$ and the fact that the root space $\mathfrak{m}^{0(10)}$ is preserved by \mathfrak{h} . This proves (iv).

We turn to (v). Let $\widehat{\mathfrak{g}}_{\leq 0} = \widehat{\mathfrak{g}}^{-2} \oplus \widehat{\mathfrak{g}}^{-1} \oplus \widehat{\mathfrak{g}}^0$ be the nonpositively graded part of $\widehat{\mathfrak{g}}$; the adjoint action of $\widehat{\mathfrak{g}}^0 \simeq \mathfrak{gl}_2(\mathbb{C}) \oplus \mathbb{C}E$ on $\widehat{\mathfrak{g}}_- = \widehat{\mathfrak{g}}^{-2} \oplus \widehat{\mathfrak{g}}^{-1}$ is given in Table 5 where $\mathbb{C}^2 = \mathfrak{g}^{-\alpha_1} \oplus \mathfrak{g}^{-(\alpha_1+\alpha_2)}$ and $(\mathbb{C}^2)^* = \mathfrak{g}^{-\alpha_3} \oplus \mathfrak{g}^{-(\alpha_3+\alpha_2)}$. We first claim that the maximal prolongation $\widehat{\mathfrak{g}}_\infty$ of $\widehat{\mathfrak{g}}_{\leq 0}$ is finite-dimensional. This is a consequence of a deep theorem of

Tanaka (see [32, Theorem 11.1] and also [32, Corollary 2, pag. 76]) based on some arguments of Serre on Spencer cohomology of Lie algebras (see [15]). In the form suitable for our purposes, this result says:

The maximal transitive prolongation $\widehat{\mathfrak{g}}_\infty$ of a fundamental and transitive \mathbb{Z} -graded Lie algebra

$$\widehat{\mathfrak{g}}_{\leq 0} = \bigoplus_{-2 \leq p \leq 0} \widehat{\mathfrak{g}}^p$$

is finite-dimensional if and only if the usual Cartan prolongation (in the sense of e.g. [29, Chapter VII]) of the linear Lie algebra

$$\mathfrak{k} = \left\{ X \in \widehat{\mathfrak{g}}^0 \mid [X, \widehat{\mathfrak{g}}^{-2}] = 0 \right\} \subset \mathfrak{gl}(\widehat{\mathfrak{g}}^{-1})$$

is finite-dimensional.

In our case $\mathfrak{k} = \mathfrak{gl}_2(\mathbb{C})$. Consider the bilinear form β on $\widehat{\mathfrak{g}}^{-1}$ given by

$$\beta(z + z^*, w + w^*) = z^*(w) + w^*(z)$$

where $z, w \in \mathbb{C}^2$ and $z^*, w^* \in (\mathbb{C}^2)^*$. Straightforward computations show:

- (i) β is symmetric and nondegenerate;
- (ii) $\mathfrak{k} \subset \mathfrak{so}(\widehat{\mathfrak{g}}^{-1}, \beta)$.

It follows that \mathfrak{k} has a trivial Cartan prolongation and hence $\dim(\widehat{\mathfrak{g}}_\infty) < +\infty$.

Let now \mathfrak{g}' be another model with $\mathfrak{g}' \supseteq \mathfrak{g}$. Note that $\mathfrak{g}'^0 = \mathfrak{g}^0$ as $\mathfrak{g}'^0 \supseteq \mathfrak{g}^0$, $\dim(\mathfrak{m}) = 7$ and $\mathfrak{n}_\# \subset \mathfrak{g}^0$. It follows that $\widehat{\mathfrak{g}}'$ is a transitive prolongation of the same $\widehat{\mathfrak{g}}'_{\leq 0} = \widehat{\mathfrak{g}}_{\leq 0}$ and hence a subalgebra of $\widehat{\mathfrak{g}}_\infty$. In particular it is finite-dimensional and $\widehat{\mathfrak{g}}' = \widehat{\mathfrak{g}}$ by [19, Theorem 3.21].

Finally (vi) follows by a direct computation using (4.1) and $\overline{\mathfrak{q}} = \widehat{\mathfrak{g}}^2 \oplus \widehat{\mathfrak{g}}^1 \oplus \widehat{\mathfrak{h}} \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{-\alpha_3} \oplus \mathfrak{g}^{-\alpha_1}$. We omit details. The theorem is proved. \square

We remark that the models \mathfrak{g} of Theorem 5.3 have a unique up to conjugation admissible Cartan subalgebra \mathfrak{h} . It is maximally noncompact only in one case, namely for $\mathfrak{g} = \mathfrak{su}(1, 3)$.

Theorem 5.1 follows from Theorem 5.3 and the following Theorem 5.4. We recall that any core $\mathfrak{m} = \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-1} \oplus \mathfrak{m}^0$, $\dim(\mathfrak{m}) = 7$, is completely determined by the complex line $\mathfrak{m}^{0(10)} \subset \mathfrak{M}^{0(10)} \simeq S^2\mathbb{C}^2$ given by $\widehat{\mathfrak{m}}^0 = \mathfrak{m}^{0(10)} \oplus \overline{\mathfrak{m}^{0(10)}}$ and that the Lie algebra $\mathfrak{n}_\#$ of the stabilizer of $\mathfrak{m}^{0(10)}$ had been described in §4.2.

Theorem 5.4. *Let \mathfrak{m} be a core, $\dim(\mathfrak{m}) = 7$, $\text{ht}(\mathfrak{m}) = 0$, given by $\mathfrak{m} = \mathfrak{m}_1$ in Table 2 or \mathfrak{m}_t with $t = \pm 1$ and $\mathfrak{m}_{\text{null}}$ in Table 4. Then the \mathbb{Z} -graded subspace $\mathfrak{g} = \bigoplus \mathfrak{g}^p$ of \mathfrak{c} with components*

$$\mathfrak{g}^p = \begin{cases} 0 & \text{for all } p < -2 \text{ and } p > 0, \\ \mathfrak{c}^p & \text{for } p = -2, -1, \\ \mathfrak{n}_\# \oplus \Re(\mathfrak{m}^{0(10)} \oplus \overline{\mathfrak{m}^{0(10)}}) & \text{for } p = 0, \end{cases}$$

is a model of type \mathfrak{m} with $\dim(\mathfrak{g})$, $\text{sgn}(\mathfrak{J})$, equivalence class $[\mathfrak{m}^{0(10)}]$ of $\mathfrak{m}^{0(10)}$ and Lie algebra $\mathfrak{n}_\#$ given by:

$\dim(\mathfrak{g})$	$\text{sgn}(\mathfrak{J})$	$[\mathfrak{m}^{0(10)}]$	\mathfrak{n}_{\sharp}
10	(2, 0)	$e_1^{10} \odot e_1^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
10	(1, 1)	$e_1^{10} \odot e_1^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
10	(1, 1)	$e_2^{10} \odot e_2^{10}$	$\mathbb{C} \oplus \mathfrak{so}_2(\mathbb{R})$
11	(1, 1)	$e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$	$\mathbb{C} \oplus (\mathbb{R} \in \mathbb{R})$

TABLE 11.

In all cases there exists an associated homogeneous CR manifold $M = G/G_o$, $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(G_o) = \mathfrak{n}_{\sharp}$, which is globally defined.

Proof. It is sufficient to show that \mathfrak{g} is a Lie subalgebra of \mathfrak{c} or, equivalently, $\widehat{\mathfrak{g}}$ a Lie subalgebra of $\widehat{\mathfrak{c}}$. Note that $\widehat{\mathfrak{g}}$ is nonpositively \mathbb{Z} -graded with $\widehat{\mathfrak{g}}^p = \widehat{\mathfrak{c}}^p$ for all $p < 0$, hence what we really need to show is only that $\widehat{\mathfrak{g}}^0 = \widehat{\mathfrak{n}}_{\sharp} \oplus \overline{\mathfrak{m}^{0(10)}} \oplus \mathfrak{m}^{0(10)}$ is a Lie subalgebra of $\widehat{\mathfrak{c}}^0$. First of all:

- (i) $[\widehat{\mathfrak{n}}_{\sharp}, \widehat{\mathfrak{n}}_{\sharp}] \subset \widehat{\mathfrak{n}}_{\sharp}$ since \mathfrak{n}_{\sharp} is a Lie algebra;
- (ii) $[\widehat{\mathfrak{n}}_{\sharp}, \mathfrak{m}^{0(10)}] \subset \mathfrak{m}^{0(10)}$ and $[\widehat{\mathfrak{n}}_{\sharp}, \overline{\mathfrak{m}^{0(10)}}] \subset \overline{\mathfrak{m}^{0(10)}}$ by the definition of \mathfrak{n}_{\sharp} ;
- (iii) $[\mathfrak{m}^{0(10)}, \overline{\mathfrak{m}^{0(10)}}] = [\overline{\mathfrak{m}^{0(10)}}, \mathfrak{m}^{0(10)}] = 0$ since $\dim_{\mathbb{C}}(\mathfrak{m}^{0(10)}) = 1$.

Finally $[\mathfrak{m}^{0(10)}, \overline{\mathfrak{m}^{0(10)}}] \subset \widehat{\mathfrak{n}}_{\sharp}$ by an explicit computation which uses (4.3) and Proposition 3.2. We only give the details for the last case of Table 11, for which $[\mathfrak{m}^{0(10)}, \overline{\mathfrak{m}^{0(10)}}] = 0$ actually holds: set

$$X = e_1^{10} \odot e_1^{10} - e_2^{10} \odot e_2^{10} - 2ie_1^{10} \odot e_2^{10}$$

and compute

$$\begin{aligned} [X, \overline{X}] &= -2ie_1^{10} \odot e_1^{01} + 2e_1^{10} \odot e_2^{01} + 2ie_2^{10} \odot e_2^{01} + 2e_2^{10} \odot e_1^{01} \\ &\quad - 2e_2^{10} \odot e_1^{01} - 2e_1^{10} \odot e_2^{01} - 2ie_2^{10} \odot e_2^{01} + 2ie_1^{10} \odot e_1^{01} \\ &= 0. \end{aligned}$$

Table 11 comes directly from Table 2 and Table 4. Finally consider the simply connected Lie group G with Lie algebra $\text{Lie}(G) = \mathfrak{g}$. It is a direct task to see that the analytic subgroup G_o of G with Lie algebra $\text{Lie}(G_o) = \mathfrak{n}_{\sharp}$ is closed in G . We omit the details. \square

Remark 5.5.

- (i) Each of the models in Table 11 satisfies property (J) since $\mathfrak{J} \in \mathfrak{n}_{\sharp}$ by definition;
- (ii) We do not know if models in Table 11 are maximal or not;
- (iii) We believe that there not exist any model associated with cores of type (B) which are not admissible and checked the conjecture when $\text{sgn}(\mathfrak{J}) = (2, 0)$. It would be interesting to understand if 7-dimensional 2-nondegenerate strongly regular CR manifolds with non-admissible cores exist or not. On this regard, we remark that for most CR-dimensions and CR-codimensions the assumption of strong regularity is quite restrictive.

6. MODELS FOR 7-DIMENSIONAL AND 3-NONDEGENERATE CR MANIFOLDS

There is a unique abstract core $\mathfrak{m} = \bigoplus \mathfrak{m}^p$ associated to 3-nondegenerate and 7-dimensional CR manifolds. It is realized inside the real contact algebra $\mathfrak{c} = \bigoplus \mathfrak{c}^p$ of degree $n = 1$ as

$$\begin{aligned} \mathfrak{m}^{-2} &= \mathfrak{c}^{-2} = \langle e^{-2} \rangle, & \mathfrak{m}^{-1} &= \mathfrak{c}^{-1} = \langle e_1, e_2 \rangle, \\ \mathfrak{m}^{0(10)} &= \mathfrak{M}^{0(10)}, & \mathfrak{m}^{1(10)} &= \mathfrak{M}^{1(10)}, \end{aligned} \quad (6.1)$$

where $\mathfrak{M}^{0(10)}$ and $\mathfrak{M}^{1(10)}$ are the $\text{ad}(\mathfrak{J})$ -eigenspaces of maximal eigenvalue $2i$ in $\widehat{\mathfrak{c}}^0$ and, respectively, $3i$ in $\widehat{\mathfrak{c}}^1$. For notational convenience, we denote the elements of the basis $\{e_1^{10}, e_1^{01}\}$ of $\widehat{\mathfrak{c}}^{-1} = S^{1,0} \oplus S^{0,1}$ simply by $z = e_1^{10}$ and $\bar{z} = e_1^{01}$, drop the symbol \odot in the expression of the symmetric products and identify each image $\text{Im}(\mu^{p|0}) \subset \mathfrak{c}^p$ with \mathbb{R} using $\mu^{p|0}(e^{-2})$ as basis (recall also the discussion after Proposition 3.2). For instance we write

$$[z, \bar{z}] = -\frac{i}{2}, \quad \mathfrak{J} = 2z\bar{z}, \quad E = -2,$$

where E is the grading element.

Theorem 6.1. *There exists a maximal model \mathfrak{g} of type (6.1) and it is unique up to isomorphisms. It is given by the 8-dimensional \mathbb{Z} -graded Lie subalgebra*

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^p$$

of the real contact algebra \mathfrak{c} of degree $n = 1$ with components

$$\mathfrak{g}^p = \begin{cases} 0 & \text{for all } p < -2 \text{ and } p > 1, \\ \mathfrak{c}^p & \text{for } p = -2, -1, \\ \Re \langle E, M, \overline{M} \rangle & \text{for } p = 0, \\ \Re \langle N, \overline{N} \rangle & \text{for } p = 1, \end{cases}$$

where $M = z^2 + z\bar{z} \in \mathfrak{M}^{0(10)}$ and $N = z^3 + 2z^2\bar{z} + z\bar{z}^2 - 3iz - 3i\bar{z} \in \mathfrak{M}^{1(10)}$. The associated terms of the Freeman sequence are

$$\begin{aligned} \mathfrak{q}_{-1} &= \mathfrak{q} = \langle z, E, M, N \rangle, & \mathfrak{q}_0 &= \langle E, M, N \rangle, \\ \mathfrak{q}_1 &= \langle E, N \rangle, & \mathfrak{q}_2 &= \mathfrak{q} \cap \overline{\mathfrak{q}} = \langle E \rangle. \end{aligned}$$

Moreover there is a 7-dimensional 3-nondegenerate homogeneous CR manifold $M = G/G_o$, $\text{Lie}(G) = \mathfrak{g}$, $\text{Lie}(G_o) = \mathbb{R}E$, which is globally defined.

Proof. We first infer some necessary conditions assuming the existence of \mathfrak{g} . We split the proof in several steps.

Step 1. The Lie subalgebra \mathfrak{g}^0 .

We claim that \mathfrak{g} does not satisfy property (\mathfrak{J}) ($\mathfrak{J} \in \mathfrak{g}^0$). Indeed in that case $\mathfrak{g}^0 = \mathfrak{c}^0$, by (6.1) and (ii)-(iv) of Definition 4.1, but any subalgebra with the same nonpositively graded part of \mathfrak{c} is of the form ([23, Proposition 3.2])

$$\begin{aligned} - \mathfrak{g} &= \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1} \oplus \mathfrak{c}^0, \\ - \mathfrak{g} &= \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1} \oplus \mathfrak{c}^0 \oplus \mu^1(\mathfrak{c}^{-1}) \oplus \mu^{2|0}(\mathfrak{c}^{-2}), \\ - \mathfrak{g} &= \mathfrak{c}^{-2} \oplus \mathfrak{c}^{-1} \oplus \mathfrak{c}^0 \oplus \bigoplus_{p>0} \mathfrak{c}^p, \\ - \mathfrak{g} &= \mathfrak{c}, \end{aligned}$$

and this is a contradiction, since $\text{ht}(\mathfrak{m}) = 1$. Hence $\dim \widehat{\mathfrak{g}}^0 = 3$ and there is a basis of $\widehat{\mathfrak{g}}^0$ of the form $\{E, z^2 + \alpha z\bar{z}, \bar{z}^2 + \bar{\alpha} z\bar{z}\}$ for some $\alpha \in \mathbb{C}$, by (ii)-(iv) of Definition 4.1. A direct computation yields

$$[z^2 + \alpha z\bar{z}, \bar{z}^2 + \bar{\alpha} z\bar{z}] = -i\bar{\alpha}z^2 - 2iz\bar{z} - i\alpha\bar{z}^2$$

but this bracket is again in $\widehat{\mathfrak{g}}^0$ if and only if $\alpha = e^{i\vartheta}$ for some $\vartheta \in [0, 2\pi)$. It follows that $\widehat{\mathfrak{g}}^0$ equals the Borel subalgebra of $\widehat{\mathfrak{c}}^0 \simeq \mathfrak{gl}_2(\mathbb{C})$ given by

$$\mathfrak{b}_\vartheta = \left\langle E, z^2 + e^{i\vartheta}z\bar{z}, \bar{z}^2 + e^{-i\vartheta}z\bar{z} \right\rangle.$$

We now see $\vartheta = 0$, up to isomorphisms of models. In view of the observation before Definition 3.5 it is enough to note that the 0-degree automorphism

$$T_\vartheta : \widehat{\mathfrak{c}}_- \longrightarrow \widehat{\mathfrak{c}}_- , \quad T_\vartheta(e^{-2}) = e^{-2} ,$$

$$T_\vartheta(z) = e^{-i\vartheta/2}z , \quad T_\vartheta(\bar{z}) = e^{i\vartheta/2}\bar{z} ,$$

satisfies the following properties:

- (i) T_ϑ is real, in the sense that $\overline{T_\vartheta(X)} = T_\vartheta(\overline{X})$ for all $X \in \widehat{\mathfrak{c}}_-$;
- (ii) T_ϑ commutes with $\text{ad}(\mathfrak{J}) : \widehat{\mathfrak{c}}_- \longrightarrow \widehat{\mathfrak{c}}_-$;
- (iii) the prolongation of T_ϑ to $\widehat{\mathfrak{c}}$ sends \mathfrak{b}_0 onto \mathfrak{b}_ϑ .

From now on $\widehat{\mathfrak{g}}^0$ is the Borel subalgebra $\mathfrak{b} = \mathfrak{b}_0 = \{E, M, \overline{M}\}$ stabilizing the line spanned by $e_1 = z + \bar{z}$.

Step 2. The space \mathfrak{g}^1 as a representation of \mathfrak{g}^0 .

We first note that the auxiliary space

$$\widetilde{\mathfrak{g}}^1 = \left\{ X \in \widehat{\mathfrak{c}}^1 \mid [X, \widehat{\mathfrak{c}}^{-1}] \subset \widehat{\mathfrak{g}}^0 \right\}$$

is a $\widehat{\mathfrak{g}}^0$ -module with $\widehat{\mathfrak{g}}^1 \subset \widetilde{\mathfrak{g}}^1$ as a submodule, as $\widehat{\mathfrak{g}}$ is a \mathbb{Z} -graded Lie algebra. Now

$$X \in \widehat{\mathfrak{c}}^1 \simeq (S^{3,0} \oplus S^{2,1} \oplus S^{1,2} \oplus S^{0,3}) \oplus (S^{1,0} \oplus S^{0,1})$$

decomposes into $X = \alpha_{30}z^3 + \alpha_{21}z^2\bar{z} + \alpha_{12}z\bar{z}^2 + \alpha_{03}\bar{z}^3 + \alpha_{10}z + \alpha_{01}\bar{z}$ and a direct computation using Proposition 3.2 shows

$$[X, z] = \frac{1}{2}(\alpha_{10} + i\alpha_{21})z^2 + \left(\frac{1}{2}\alpha_{01} + i\alpha_{12}\right)z\bar{z} + \frac{3}{2}i\alpha_{03}\bar{z}^2 + \frac{1}{2}i\alpha_{01} ,$$

$$[X, \bar{z}] = -\frac{3}{2}i\alpha_{30}z^2 + \left(\frac{1}{2}\alpha_{10} - i\alpha_{21}\right)z\bar{z} + \frac{1}{2}(\alpha_{01} - i\alpha_{12})\bar{z}^2 - \frac{1}{2}i\alpha_{10} .$$

It turns out that these brackets are in $\widehat{\mathfrak{g}}^0$ if and only if the following linear system of equations is satisfied:

$$\alpha_{10} - 2i\alpha_{21} = \alpha_{01} - i\alpha_{12} - 3i\alpha_{30} ,$$

$$\alpha_{01} + 2i\alpha_{12} = 3i\alpha_{03} + \alpha_{10} + i\alpha_{21} .$$

In other words

$$\widetilde{\mathfrak{g}}^1 = (z + \bar{z}) \odot \mathfrak{b} \bigoplus \left\langle z^2\bar{z} + z\bar{z}^2 + \frac{i}{2}z - \frac{i}{2}\bar{z} \right\rangle$$

with a basis of the form $\{N, \bar{N}, V, W\}$ where

$$\begin{aligned} V &= z^2 \bar{z} + z \bar{z}^2 + \frac{i}{2} z - \frac{i}{2} \bar{z}, \\ W &= z + \bar{z} \\ N &= (z + \bar{z}) \odot (z^2 + z \bar{z}) - 3i(z + \bar{z}) \\ &= z^3 + 2z^2 \bar{z} + z \bar{z}^2 - 3iz - 3i\bar{z}. \end{aligned}$$

The element N is characterized by the following property: it is the unique non-trivial element in $\hat{\mathfrak{g}}^1 \cap \mathfrak{u}$ which commutes with its conjugate (we omit the long but straightforward proof of this fact, based again on Proposition 3.2). We also note that $\pi_{\mathfrak{M}^{1(10)}}(V) = \pi_{\mathfrak{M}^{1(01)}}(V) = \pi_{\mathfrak{M}^{1(10)}}(W) = \pi_{\mathfrak{M}^{1(01)}}(W) = 0$.

Now (iii)-(iv) of Definition 4.1 say that $\hat{\mathfrak{g}}^1 \cap \bar{\mathfrak{u}}$ is at least one-dimensional, including an element of the form

$$\bar{N}_{\alpha\beta} = \bar{N} + \alpha V + \beta W,$$

for some $\alpha, \beta \in \mathbb{C}$. Clearly $N_{\alpha\beta} \in \hat{\mathfrak{g}}^1 \cap \mathfrak{u}$ too. From $\text{ad}(\mathfrak{J})$ -equivariance and Proposition 3.2, we get for all $\delta, \gamma \in \mathbb{C}$

$$\begin{aligned} \pi_{\mathfrak{M}^{2(10)}}[N_{\alpha\beta}, \gamma V + \delta W] &= \pi_{\mathfrak{M}^{2(10)}}[z^3, (\delta + \frac{i}{2}\gamma)z + \gamma z^2 \bar{z}] \\ &= (\frac{1}{2}\delta - \frac{5}{4}i\gamma)z^4, \\ \pi_{\mathfrak{M}^{2(01)}}[\bar{N}_{\alpha\beta}, \gamma V + \delta W] &= \pi_{\mathfrak{M}^{2(01)}}[\bar{z}^3, (\delta - \frac{i}{2}\gamma)\bar{z} + \gamma z \bar{z}^2] \\ &= (\frac{1}{2}\delta + \frac{5}{4}i\gamma)\bar{z}^4, \end{aligned}$$

where $\pi_{\mathfrak{M}^{2(10)}} : \hat{\mathfrak{c}}^2 \rightarrow \mathfrak{M}^{2(10)}$ and $\pi_{\mathfrak{M}^{2(01)}} : \hat{\mathfrak{c}}^2 \rightarrow \mathfrak{M}^{2(01)}$ are the projections onto the $\text{ad}(\mathfrak{J})$ -eigenspaces of extremal eigenvalues $\pm 4i$ in $\hat{\mathfrak{c}}^2$. But $\text{ht}(\mathfrak{m}) = 1$ and therefore an element $\gamma V + \delta W$ belongs to $\hat{\mathfrak{g}}^1$ if and only if $\gamma = \delta = 0$. In other words we obtained $\hat{\mathfrak{g}}^1 = \langle N_{\alpha\beta}, \bar{N}_{\alpha\beta} \rangle$. We now prove $\alpha = \beta = 0$.

By $\text{ad}(\mathfrak{J})$ -equivariance and $[N, \bar{N}] = 0$, we get

$$\begin{aligned} \pi_{\mathfrak{M}^{2(10)}}[N_{\alpha\beta}, \bar{N}_{\alpha\beta}] &= \pi_{\mathfrak{M}^{2(10)}}[N, \bar{N}_{\alpha\beta}] \\ &= \pi_{\mathfrak{M}^{2(10)}}[N, (\beta + \frac{i}{2}\alpha)z + \alpha z^2 \bar{z}] \\ &= \pi_{\mathfrak{M}^{2(10)}}[z^3, (\beta + \frac{i}{2}\alpha)z + \alpha z^2 \bar{z}] \\ &= (\frac{1}{2}\beta - \frac{5}{4}i\alpha)z^4, \end{aligned}$$

forcing $2\beta = 5i\alpha$. On the other hand

$$\begin{aligned} [2M, \bar{N}_{\alpha\beta}] &= -2i(1 + \alpha)z^3 - i(7 + 3\alpha)z^2 \bar{z} - i(8 + \alpha)z \bar{z}^2 - 3i\bar{z}^3 \\ &\quad + (3 + 2\alpha)\bar{z} + (3 + \alpha)z \end{aligned}$$

and the condition $[2M, \bar{N}_{\alpha\beta}] \in \hat{\mathfrak{g}}^1$ is equivalent to the following system of non-linear equations on $\alpha \in \mathbb{C}$,

$$2\alpha + \bar{\alpha} + \alpha\bar{\alpha} = 0, \quad 2\alpha + 6\bar{\alpha} + 6\alpha\bar{\alpha} = 0, \quad 2\alpha - 4\bar{\alpha} - 4\alpha\bar{\alpha} = 0.$$

The unique solution of this system is $\alpha = 0$, hence $\beta = 0$ and $\hat{\mathfrak{g}}^1 = \langle N, \bar{N} \rangle$.

It is not difficult to see now that $\widehat{\mathfrak{g}} := \widehat{\mathfrak{c}}^{-2} \oplus \widehat{\mathfrak{c}}^{-1} \oplus \widehat{\mathfrak{g}}^0 \oplus \widehat{\mathfrak{g}}^1$ is a complex Lie subalgebra of $\widehat{\mathfrak{c}}$ and $\mathfrak{g} = \Re(\widehat{\mathfrak{g}})$ a model of type (6.1). The uniqueness and maximality of \mathfrak{g} are a consequence of the next step.

Step 3. The space \mathfrak{g}^p is trivial for all $p \geq 2$.

We show that the auxiliary subspace $\widetilde{\mathfrak{g}}^2$ of $\widehat{\mathfrak{c}}^2$ determined by

$$\begin{aligned} \widetilde{\mathfrak{g}}^2 &\subset (S^{3,1} \oplus S^{2,2} \oplus S^{1,3}) \oplus (S^{2,0} \oplus S^{1,1} \oplus S^{0,2}) \oplus S^{0,0} \subset \widehat{\mathfrak{c}}^2 \\ [\widetilde{\mathfrak{g}}^2, \widehat{\mathfrak{c}}^{-1}] &\subset \widehat{\mathfrak{g}}^1 \end{aligned}$$

is trivial. This clearly implies $\widehat{\mathfrak{g}}^2 = 0$ and also $\widehat{\mathfrak{g}}^p = 0$ for all $p > 2$, by the transitivity of $\widehat{\mathfrak{c}}$. Now any $X \in \widetilde{\mathfrak{g}}^2$ satisfies $[X, e^{-2}] \subset \widehat{\mathfrak{g}}^0$ and it is therefore of the form

$$\begin{aligned} X &= \alpha_{31} z^3 \bar{z} + \alpha_{22} z^2 \bar{z}^2 + \alpha_{13} z \bar{z}^3 \\ &\quad + \alpha_{20} z^2 + (\alpha_{20} + \alpha_{02}) z \bar{z} + \alpha_{02} \bar{z}^2 + \alpha_{00}, \end{aligned}$$

for some $\alpha_{31}, \dots, \alpha_{00}$ in \mathbb{C} ; from this and Proposition 3.2 we get

$$\begin{aligned} [X, z] &= \left(\frac{i}{2}\alpha_{31} + \frac{1}{2}\alpha_{20}\right)z^3 + \left(i\alpha_{22} + \frac{1}{2}\alpha_{02} + \frac{1}{2}\alpha_{20}\right)z^2 \bar{z} \\ &\quad + \left(\frac{3}{2}i\alpha_{13} + \frac{1}{2}\alpha_{02}\right)z \bar{z}^2 + \left(\frac{i}{2}\alpha_{02} + \frac{i}{2}\alpha_{20} + \frac{1}{2}\alpha_{00}\right)z + i\alpha_{02} \bar{z}, \\ [X, \bar{z}] &= \left(-\frac{3}{2}i\alpha_{31} + \frac{1}{2}\alpha_{20}\right)z^2 \bar{z} + \left(-i\alpha_{22} + \frac{1}{2}\alpha_{02} + \frac{1}{2}\alpha_{20}\right)z \bar{z}^2 \\ &\quad + \left(-\frac{i}{2}\alpha_{13} + \frac{1}{2}\alpha_{02}\right)\bar{z}^3 - i\alpha_{20}z + \left(-\frac{i}{2}\alpha_{02} - \frac{i}{2}\alpha_{20} + \frac{1}{2}\alpha_{00}\right)\bar{z}. \end{aligned}$$

The claim $X = 0$ follows from the fact that conditions “ $[X, z]$ proportional to N ” and “ $[X, \bar{z}]$ proportional to \bar{N} ” are equivalent to an homogeneous linear system of six equations and six indeterminates which is nondegenerate.

Step 4. The last claims.

The terms of the Freeman sequence follow from a direct computation. The existence of a globally well-defined associated homogeneous CR manifold is given by considering the simply connected Lie group G with Lie algebra $\text{Lie}(G) = \mathfrak{g}$ and the closed subgroup G_o with one-dimensional Lie algebra $\text{Lie}(G_o) = \mathbb{R}E$ spanned by the grading element. We omit details. \square

APPENDIX A. PROOF OF PROPOSITION 4.8.

We first consider the reducible representation (4.10) of \widetilde{K} . The associated orbits $\widetilde{K} \cdot z$ split in two types, according to whether the real and imaginary components x and y of $z = x + iy$ are linearly dependent over \mathbb{R} or not.

In the first case $\widetilde{K} \cdot z \simeq \widetilde{K}/\widetilde{H}$ where $\widetilde{H} \simeq \text{O}_2(\mathbb{R})$ and a complete set of representatives for these orbits is

$$\left\{ z_t = (1 + it)\epsilon_1 \mid t \in \mathbb{R} \right\} \cup i \cdot \epsilon_1. \quad (\text{A.1})$$

In the second case $\widetilde{K} \cdot z \simeq \widetilde{K}/\widetilde{H}$ where $\widetilde{H} \simeq \mathbb{Z}_2$ and the representative set is parametrized by the upper half plane

$$\text{H} = \left\{ (t_1, t_2) \mid t_2 > 0 \right\}$$

and given by

$$\left\{ z_{t_1, t_2} = (1 + it_1)\epsilon_1 + (it_2)\epsilon_2 \mid (t_1, t_2) \in \text{H} \right\}. \quad (\text{A.2})$$

Recall that the actions of \tilde{K} and \mathbb{C}^\times commute. In particular any $c = e^{i\vartheta}$ sends an orbit $\tilde{K} \cdot z$ of type \tilde{H} onto an orbit of the same type.

Using (A.1) we check that all orbits of type $\tilde{H} \simeq \mathrm{O}_2(\mathbb{R})$ are related and glue into a single $K_\#$ -orbit, say $K_\# \cdot \epsilon_1$. In particular any $e^{i\vartheta}$ acts also on

$$\mathcal{U} = V^\times \setminus K_\# \cdot \epsilon_1 = \bigcup_{(t_1, t_2) \in \mathcal{H}} \tilde{K} \cdot z_{t_1, t_2},$$

the collection of all orbits of type $\tilde{H} \simeq \mathbb{Z}_2$. More precisely any $z = x + iy \in \mathcal{U}$ belongs to the orbit $\tilde{K} \cdot z_{t_1, t_2}$ where

$$t_1 = \frac{\langle x, y \rangle}{\langle x, x \rangle}, \quad (t_2)^2 = \frac{\langle y - t_1 x, y - t_1 x \rangle}{\langle x, x \rangle}. \quad (\text{A.3})$$

This follows from a check with $z = z_{t_1, t_2}$ and from the observation that the r.h.s. of the two identities are constants on the orbits. In other words there is an action of S^1 on \mathcal{H} with the property that $e^{i\vartheta} \cdot (t_1, t_2) = (t'_1, t'_2)$ if and only if the associated orbits $\tilde{K} \cdot z_{t_1, t_2}$ and $\tilde{K} \cdot z_{t'_1, t'_2}$ are subsets of a $K_\#$ -orbit.

A computation using (A.3) and $z' = e^{i\vartheta} \cdot z \in \tilde{K} \cdot z_{t'_1, t'_2}$ yields the explicit expression of this action

$$t'_1 = \frac{\frac{1}{2}(1 - t_1^2 - t_2^2) \sin 2\vartheta + t_1 \cos 2\vartheta}{(t_1 \sin \vartheta - \cos \vartheta)^2 + (t_2 \sin \vartheta)^2}, \quad (\text{A.4})$$

$$(t'_2)^2 = \frac{(1 + t_1^2)^2 \sin^2 \vartheta + t_2^2 (t_1 \sin \vartheta + \cos \vartheta)^2}{(t_1 \sin \vartheta - \cos \vartheta)^2 + (t_2 \sin \vartheta)^2}, \quad (\text{A.5})$$

and a continuity argument in ϑ from 0 to $\pi/2$ in (A.4) says that (t_1, t_2) is always on the S^1 -orbit of some (t'_1, t'_2) with $t'_1 = 0$. Exploiting (A.4)-(A.5) with now $t_1 = t'_1 = 0$ we finally get $(0, 1]$ as set of representatives for \mathcal{H}/S^1 .

Summarizing: for $t \in [0, 1]$ the $K_\#$ -orbits $K_\# \cdot z_{0, t} = K_\# \cdot (\epsilon_1 + it\epsilon_2)$ are pairwise-disjoint and their union is the entire V^\times . This is the first claim. Using (4.9) one checks finally that $N_\#$ is isomorphic to $\mathbb{C}^\times \cdot \mathrm{O}_2(\mathbb{R})$ if $t = 0$, $\mathbb{C}^\times \cdot \mathbb{Z}_2$ if $0 < t < 1$, $\mathbb{C}^\times \cdot \mathrm{SO}_2(\mathbb{R})$ if $t = 1$. This implies the last claim. \square

APPENDIX B. PROOF OF PROPOSITION 4.10.

It is similar to Proposition 4.8 and we only give the main steps. As before, we first split the orbits in two types. In the first case any orbit is equivalent to one displayed in Table 3 and a complete set of representatives is

$$\begin{aligned} \left\{ z_t = (1 + it)\epsilon_1 \right\} \cup i \cdot \epsilon_1, \quad & \left\{ w_t = (1 + it)\epsilon_3 \right\} \cup i \cdot \epsilon_3 \\ \text{and} \quad & \left\{ u_t = (1 + it)(\epsilon_1 + \epsilon_3) \right\} \cup i \cdot (\epsilon_1 + \epsilon_3), \end{aligned} \quad (\text{B.6})$$

where $t \in \mathbb{R}$. The associated stabilizers \tilde{H} are also in Table 3. In the second case we first use \tilde{K} to fix x equal to ϵ_1 if space-like (resp. $\epsilon_1 + \epsilon_3$ if null, ϵ_3 if time-like) and then the stabilizer of x in \tilde{K} to fix y . This gives the following six different types of representatives:

- (i): $\epsilon_1 + i(t_1\epsilon_1 + t_2\epsilon_2)$ where $t_1 \in \mathbb{R}$ and $t_2 \neq 0$;
- (ii): $\epsilon_1 + \epsilon_3 + i(t(\epsilon_1 + \epsilon_3) \pm (\epsilon_1 - \epsilon_3))$ where $t \in \mathbb{R}$;

- (iii): $\epsilon_1 + \epsilon_3 \pm i\epsilon_2$;
- (iv): $\epsilon_1 + i(t_1\epsilon_1 + t_3\epsilon_3)$ where $t_1 \in \mathbb{R}$ and $t_3 > 0$,
- (v): $\epsilon_1 + i(t\epsilon_1 \pm (\epsilon_2 + \epsilon_3))$ where $t \in \mathbb{R}$,
- (vi): $\epsilon_3 + i(t_1\epsilon_1 + t_3\epsilon_3)$ where $t_1 > 0$ and $t_3 \in \mathbb{R}$.

The stabilizer \tilde{H} is always trivial, except in case (i) where $\tilde{H} \simeq \mathbb{Z}_2$.

Any $e^{i\vartheta}$ sends an orbit $\tilde{K} \cdot z$ of type \tilde{H} onto an orbit of the same type. We first focus on representatives (B.6). All \tilde{K} -orbits with $\tilde{H} \simeq \text{SO}^+(1, 1) \cup \text{SO}^+(1, 1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ are related and glue into $K_{\#} \cdot \epsilon_1$. Similarly those of type $\tilde{H} \simeq \text{SO}_2(\mathbb{R})$ (resp. $\tilde{H} \simeq \mathbb{R}^+ \ltimes \mathbb{R}$) glue into $K_{\#} \cdot \epsilon_3$ (resp. $K_{\#} \cdot (\epsilon_1 + \epsilon_3)$).

The collection of all \tilde{K} -orbits of type $\tilde{H} \simeq \mathbb{Z}_2$ is stable under any $e^{i\vartheta}$ and the representative of $\tilde{K} \cdot e^{i\vartheta} \cdot (\epsilon_1 + i(t_1\epsilon_1 + t_2\epsilon_2))$ is $(\epsilon_1 + i(t'_1\epsilon_1 + t'_2\epsilon_2))$ where

$$t'_1 = \frac{t_1(\cos^2 \vartheta - \sin^2 \vartheta) + \frac{1}{2}(1 - t_1^2 - t_2^2) \sin 2\vartheta}{\cos^2 \vartheta + (t_1^2 + t_2^2) \sin^2 \vartheta - t_1 \sin 2\vartheta}, \quad (\text{B.7})$$

$$t'_2 = \frac{t_2}{\cos^2 \vartheta + (t_1^2 + t_2^2) \sin^2 \vartheta - t_1 \sin 2\vartheta}. \quad (\text{B.8})$$

A continuity argument implies that (t_1, t_2) is always on the S^1 -orbit of some (t'_1, t'_2) with $t'_1 = 0$ and hence, using (B.7)-(B.8) with $t_1 = t'_1 = 0$, that

$$\left\{ \epsilon_1 + it\epsilon_2 \mid -1 \leq t \leq 1, t \neq 0 \right\}$$

parametrizes the representatives of the $K_{\#}$ -orbits $K_{\#} \cdot (\epsilon_1 + it\epsilon_2)$.

Finally the collection of \tilde{K} -orbits with $\tilde{H} = \{1\}$, i.e. with representative as in (ii)-(vi) above, is stable under any $e^{i\vartheta}$. Applying appropriate $e^{i\vartheta}$ to any representative in (iv)-(vi) we can always reach a $z = x + iy$ with a null real component x . In other words any orbit with representative in (iv)-(vi) is S^1 -related with at least one orbit with representative in (ii) or (iii).

By a similar argument we also see that the representatives in (iii) are representatives of the associated $K_{\#}$ -orbits too whereas the representatives of $K_{\#}$ -orbits as in (ii) are given by $\epsilon_1 + \epsilon_3 + i(t(\epsilon_1 + \epsilon_3) + (\epsilon_1 - \epsilon_3))$, $t \in \mathbb{R}$.

Summarizing the results gives the first claim. Using (4.9) we finally see

$$N_{\#} = \begin{cases} \mathbb{C}^{\times} \cdot (\text{SO}^+(1, 1) \cup \text{SO}^+(1, 1) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}) & \text{for } \epsilon_1, \\ \mathbb{C}^{\times} \cdot \text{SO}_2(\mathbb{R}) & \text{for } \epsilon_1 \pm i\epsilon_2 \text{ and } \epsilon_3, \\ \mathbb{C}^{\times} \cdot (\mathbb{R}^+ \ltimes \mathbb{R}) & \text{for } \epsilon_1 + \epsilon_3, \\ \mathbb{C}^{\times} \cdot \mathbb{Z}_2 & \text{for } \epsilon_1 + it\epsilon_2 \text{ where } -1 < t < 1, t \neq 0, \\ \mathbb{C}^{\times} & \text{for } \epsilon_1 + \epsilon_3 \pm i\epsilon_2 \text{ and all } \epsilon_1 + \epsilon_3 + i(t(\epsilon_1 + \epsilon_3) + (\epsilon_1 - \epsilon_3)), \end{cases}$$

which gives the last claim. \square

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